Innovative Damage and Constitutive Modeling of Fiber Reinforced Cementitious Composites Subjected to Earthquake Loads

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Innovative Damage and Constitutive Modeling of Fiber Reinforced Cementitious Composites Subjected to Earthquake Loads

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This one-year research project focuses on innovative micromechanical modeling of multiphase fiber reinforced concrete. This research is an extension of previous research conducted by the CUREe Principal Investigator in collaboration with research engineers at Kajima Technical Research Institute. In particular, this research investigates: (a) the elastic stress fields arising from the single fiber pullout system by using both the analytical and finite element methods, and (b) the local interactions and effective transverse elastic properties of two-dimensional two-phase unidirectionally aligned, randomly located fiber reinforced concrete. The research conducted under this CUREe-Kajima project enables us to accurately predict the local and overall mechanical behavior and performance of advanced high-strength fiber reinforced concrete for use in buildings and infrastructure systems.

1. Research Objectives

The overall goals of the proposed research are to propose advanced micromechanics-based constitutive and damage models (microcracking and fiber pull-out) of high-strength fiber reinforced cementitious composites for use in structures and infrastructures. The proposed research and expected results will render new micromechanics- and microstructure-based constitutive and damage models which are capable of accurately explaining and predicting mechanical behavior of high-strength cementitious composites subjected to natural hazards such as earthquake and wind loadings. Furthermore, the proposed research will be able to predict and suggest new ways to improve performance and economy of FRC material systems and structures.

In particular, the objectives of the proposed research are as follows.

(1) To develop innovative predictions of effective elastic and elastic-damage properties of two-dimensional two-phase and three-phase fiber reinforced mortar and concrete.

(2) To investigate micromechanics of interfacial debonding and fiber pull-out (between fibers and the mortar matrix) in three-dimension for unidirectional long fiber and random short fiber reinforced mortar and concrete.
2. Major Research Accomplishments

There are several major research accomplishments as a result of this project. For convenience, these are categorized into three parts.

Part I. Elastic stress fields arising from the single fiber pullout problem: Numerical results

Part II. Elastic stress fields arising from the single fiber pullout problem: Analytical solutions

Part III. Micromechanics and effective transverse elastic moduli of fiber reinforced composites
Chapter 1

Introduction

1.1 Motivation

The use of fiber reinforced composite material has rapidly increased in different fields during the past few decades. Recent studies of fiber reinforced materials have shown that the matrix materials can be toughened by adding the fiber reinforcements. It is generally accepted that the major contribution of the fiber reinforcements is to improve the tensile strain capacity of the matrix material. Figure 1.1 is an example of such improvement. The stress-displacement curve shows clearly that the ductility of mortar matrix had been dramatically improved by even only adding very small amount of fiber reinforcements. The further application of these fiber reinforced materials can fundamentally change the behavior of structures such as demonstrated in Figure 1.2. The bending capacity of the compact, reinforced composite (made of cement, silica fume, and steel microfibers) is an order of magnitude higher than that of conventional reinforced concrete. The other advantage of fiber reinforced material is that different type of fiber reinforcement can be used to achieve different purpose. For example, glass fibers used in the cement-based composite are generally alkali resistant. It is no doubt that an attractive material like fiber reinforced composite is will be more and more widely used in the future. The development of this
new class of composite depends on better understanding of the toughening mechanism of the fiber reinforced materials.

The toughening mechanisms of fiber reinforced composites have been well recognized to be fiber bridging and fiber/matrix interface debonding, Figure 1.3. As a crack propagate in the material, the fibers behind the crack tip will provide a bridging force to against the crack opening. Also, when the crack tip encountered the fiber, it will deflected along the fiber/matrix interface effectively blunting the crack tip. This is the interface debonding mechanism. In the sense, it is similar to the concept of drilling a hole at crack tip to prevent crack propagation. The debonded interface can also provide part of the bridging force through the interface frictional force. In the energy point of view, the interface debonding and frictional sliding can both absorb more energy which reflects in the stress-strain curve is the area under the curve. Thus the fiber reinforced material can have more ductility. If the bond between the interface is too strong, the interface will not debond, i.e., the crack will not deflect along the interface and will keep propagate in the matrix. The debonding mechanism will not have any effect here. On the other hand, the frictional force at the debonded interface will depend on the interface frictional coefficient. In other words, the mechanical behavior of the fiber reinforced composites is mainly controlled by the fiber/matrix interface properties, namely the interface bond strength and frictional coefficient. Several methods have been developed to determine the fiber/matrix interface properties, e.g. the fiber pull out test, the fiber push out (or indentation) test, the slice compression test and the fiber fragmentation test. Among these methods, the single fiber pull out test is the most popular mechanical test technique. The stress distributions in the fiber/matrix interface should be known in order to better characterize fiber/matrix interface behavior thus evaluate interface properties through the experimental data.

A significant effort has been put into the theoretical analysis of the pull-out problem. Muki and Stemberg(1969,1970) studied the stress transfer from the embedded into the surrounding matrix and gave a solution in the form of a Fredholm integral equation which is
consistent with the linear theory of elasticity. The idealization made in their paper is that the fiber is considered as a one-dimensional continuum embedded in an elastic half space. Luk and Keer (1979) studied similar problem but idealized the fiber as a rigid rod. They analyzed the case with extremely small fiber length to diameter ratios and gave a solution utilizing a Hankel integral transformation. Greszczuk (1969), Lawrence (1972), Piggot (1980), Gopalaratnam and Shah (1987), Hsueh (1988, 1990) had derived different approximate solutions based on the so called shear-lag theory. The shear-lag theory is originally proposed by Cox (1952) and have been utilized by many researchers. Actually most of the recent studies of the single fiber pullout problem done were based on this theory. However it is found that the solutions based on the shear-lag theory can not give an appropriate description of the stress field arising in the neighborhood of the fiber entry and embedded end. Atkinson et al. (1982) and Marotzke (1994) have used finite element method to calculate the elastic stress fields along the fiber/matrix interface in the single fiber pull out problem. But they were all only focus on certain cases. The purpose of this study is to study the elastic stress fields arising in the single fiber pull out test through both linear elastic finite element analysis and theoretical analysis. By examining these elastic stress fields, one can get some general ideas about the effects of different parameters, e.g., the fiber embedded length, the ratio of fiber/matrix diameter, and the ratio of fiber/matrix Young's modulus. Based on which, the assumptions of the existing analytical models were discussed. Some suggestions about utilizing the single fiber pull out test were also made.
1.2 References


Volumn fraction $v_f = 1.5\%$

Volumn fraction $v_f = 1.0\%$

Plain mortar matrix

Fig. 1.1 Stress and displacement curves of plain mortar matrix and steel-fiber-reinforced composites. (Shah 1991)
Fig. 1.2 Compacted reinforced composite beam made of cement-based matrix and microfibers. (Shah 1991)
Fig. 1.3 Illustration of the toughening mechanisms

Fiber bridging

Matrix

Fiber

Interface debonding
Chapter 2

Elastic Stress Fields Arising in the
Single Fiber Pullout System :
Numerical Analysis

2.1 Introduction

In the study of the mechanical behavior of fiber reinforced composite, it is well recognized that the major toughening mechanisms are fiber bridging and fiber/matrix interface debonding. And these two mechanisms are critically influenced by the fiber/matrix interface properties, i.e., the fiber/matrix interface bond strength and frictional coefficient between the debonded interface. In other words, the fiber/matrix interface is the most important issue in studying the mechanical behavior of the fiber reinforced composites. Different methods have been developed to determine the fiber/matrix interface properties, e.g. the fiber pull out test, the fiber push out (or indentation) test, the slice compression test and the fiber fragmentation test. Among these methods, the single fiber pull out test is the most popular mechanical test technique because of its simplicity and versatility. Usually the single fiber pullout test will give the experimental data which represents a stress(applied load)-displacement curve. Figure 2.1 illustrate a typical load-displacement curve from a single
fiber pullout test. A single fiber pull out model is then needed to evaluate the interface properties. The initial linear region of the pullout load versus slip curve is followed by a monotonic change in the slope, which characterizes the initiation and propagation of interface debonding. The interface will totally debond after the maximum load is reached and then the postpeak behavior is governed by the frictional sliding. A significant effort has been put into the characterization of the interface properties through the experimental data, Hsueh(1990), Li, Mobasher, and Shah(1991), Meretz, Auersh, Marotzke, Schulz, and Hampe(1993), Kuntz, Schlapschi, Meier, and Grathwohl(1994), Cordes, and Daniel(1995). Most of the works done were based on the shear-lag model which has been found can not give an appropriate description of the stress field arising in the neighborhood of the fiber. All the solutions based on shear lag theory describe the interfacial stresses as hyperbolic sine or cosine functions. The hyperbolic sine or cosine functions decreases monotonically up to the fiber embedded end. Thus the stresses concentration at the fiber embedded end can not be predicted. This means that the model based on shear lag model will be lack of ability in capturing the two-way debonding phenomenon which had been discussed by Atkinson et al.(1982), Leung and Li(1991). Atkinson et al.(1982), Marotzke(1994), Cherepanov and Esparragoza(1995) have used finite element method to calculate the elastic stress fields along the fiber/matrix interface in the single fiber pull out problem. But they were all only focus on certain cases. The purpose of this work is to study the influences of different elastic geometrical and material parameters on stress fields arising in the single fiber pull out system by using linear elastic finite element analysis. By examining these elastic stress fields, one can get some general ideas about the effects on the stress fields arising in the single fiber pull out system of chosen parameters. From which one can make some reasonable assumptions to improve the analytical model and have some idea about the limitation of the existing model. Thus more accurate evaluation of the interface properties from the single fiber pull out test data can be achieved. The parameters considered in this study are the fiber embedded length, the ratio of fiber/matrix diameter, and the ratio of fiber/matrix Young’s modulus.
2.2 The Finite Element Model

The single fiber pull out model consists of a single fiber embedded in a cylinder of matrix as shown in Figure 2.2. The fiber cross section is assumed to be a circle. According to the axisymmetric property of this idealized model, the finite element analysis can be reduced to a 2-D, plane strain, problem. A four-node, axisymmetric, quadrilateral element was used in this study. The geometric configuration of the finite element model and the boundary conditions are shown in Figure 2.3. Since the purpose of this study is to examine the influences of different aspect ratios, nondimensional geometric was used in the finite element model. The radius of the fiber is 0.5 and the fiber embedded length is varying from 40% to 80% of the matrix length. Different matrix radius and length were used. The fiber/matrix diameter ratio was set to be 33.333%, 20%, 10%, and 5%. The length of 25, 50, and 100 for the matrix were considered in this study. The free fiber end is subjected to unit axial stresses which is uniformly distributed over the cross section. It should be noted that a part of fiber with the length equal to the fiber diameters outside the matrix is taken into account in the finite element model. This is not only consistent with the configuration of single fiber pullout model but also ensure an unrestricted variation of the stress field near the fiber entry which is expected to be influenced by the stress concentrations.

It is well known that the singular stress field can arise at the corners of boundaries or interfaces between phases with different elastic constants within the framework of the linear elastic theory, Hein and Erdogan (1971), Rao(1971). The stress singularity which means an unbounded stress at the corners of boundaries or interfaces can not occur in real materials. The theoretical problem of singularity will turn out to be the stress concentration problem in the finite element analysis. It should be pointed out that the element used in this study is not capable of representing the singularities, i.e., the magnitude of the stress concentration will depends on the element mesh. However, reasonable solution can be obtained in the vicinity
of the singularity region if the element mesh is sufficiently small, Tong and Pian(1973), Whitcomb et al(1982). This implies that we can get reliable trend of the elastic stress fields by using large number of elements in the finite element analysis. The element mesh used in this study contains 45 elements in the radial direction and 80-140 elements in the axial direction, depending on the matrix cylinder length. Very fine mesh were used in the vicinity of the fiber entry and the fiber end, where stress concentration will occur. The size of the smallest elements are about 1/15 of the fiber radius in each direction.

Both fiber and matrix are assumed to be linear elastic and isotropic. The fiber/matrix interface is assumed to be perfectly bonded. Two sets of material properties were used in this study. In the first set, the matrix is polycarbonate, $E_m = 2.3 \text{ Gpa}$ and $\nu_m = 0.38$, and the fiber is E-glass fiber, $E_f = 73.5 \text{ Gpa}$ and $\nu_f = 0.18$. This set of material properties were the same as used in Marotzke(1994). In the second set of material properties, the Young’s modulus of the matrix, $E_m$, is 28.8 Gpa and the Poisson’s ratio, $\nu_m$, is 0.2. The value chosen here is between 10 – 45 Gpa which is the range for the Young’s modulus of cement. The Young’s modulus of the fiber in the second set of material properties are varying according to the $E_f/E_m$ ratio while the Poisson’s ratio, $\nu_f$, is 0.2. The $E_f/E_m$ ratios considered in this study are 5, 10, 20, and 50. Since we used the same element mesh in the vicinity of the singularities in each analysis, the numerical peak values of the stresses in each analysis can be compared in order to qualitatively examine the influence of the chosen parameters on the stress concentrations.
2.3 Results and Discussion

2.3.1 The stress fields

Analyzing the system with first set of material properties and let fiber diameter equal to 1 while the embedded length is 15 and matrix length is 25, we obtained similar elastic stress fields as in Marotzke (1994). The elastic stress distribution along the axial direction is shown in Figure 2.4 through Figure 2.7. Although the stress concentration level is different, it occurs at both fiber entry and embedded end as expected. Most of the analytical models of the single fiber pullout problem based on the shear-lag theory describe the interfacial shear stress as hyperbolic cosine functions, which decreases monotonically up to the fiber embedded end. Obviously the fact of stress concentration at the fiber embedded end can not be predicted by these models.

The axial stresses decay from the fiber entry to the fiber embedded end and are almost constant with respect to the radial coordinate, except in the zones near the fiber entry and the embedded end. Where the axial stress increase from the fiber center up to the fiber/matrix interface. The detail is shown in Figure 2.4.2. The decay of the fiber axial stresses shown in Figure 2.4.1 reveals the fact that the force transferred into the matrix from the fiber. The axial stresses in the matrix are also shown in the figure. It is also almost constant with respect to the radial coordinate except in the region close to the fiber entry. However the values are much lower than those in the fiber. Theoretically the difference will depend on the difference of Young’s moduli between fiber and matrix. Similar behavior can be found at the fiber embedded end, 2.4.3. The distribution of the axial stresses reveals that the transfer of the fiber force into the matrix mainly takes place by interfacial shear stress. Only a small amount of fiber force is transferred through the fiber embedded end since the values of remaining axial stresses in the fiber are very small. Actually many models did not
consider the stress transfer through the bottom of fiber embedded end, i.e., the boundary at bottom of the fiber embedded end was assumed to be free.

The radial stresses are displayed in Figure 2.5. The radial stresses decrease rapidly at the fiber entry form a maximum tensile value to a very small compressive value, which is nearly vanished. Just before the fiber embedded end, the radial stress reach a sharp compressive maximum value which is smaller comparing with that at the fiber entry. Beyond the fiber embedded end, the radial stress change sign and reach a second tensile maximum. Figure 2.6 shows the distribution of the hoop stress, which has similar behavior to that of the radial stresses except the stress concentration value is smaller. The hoop stress are usually not taken into account when considering the failure of the fiber/matrix interface in the single fiber pullout problem.

The distribution of shear stresses is given in Figure 2.7.1. Not like the axial stresses, the shear stresses reach their maximum level in the interface, shown in Figure 2.7.2. They decrease from a maximum value at the fiber/matrix interface to the fiber center and the free matrix surface, where the values approach zero, with respect to the radial coordinate. Since the interfacial shear stresses govern the stress transfer from fiber to matrix, most people treat the interface debonding as mode II fracture, Hutchinson and Jensen (1990), Bazant and Desmorat (1994). In this point of view, the interfacial shear stresses will play an very important role in the interface debonding process. This will be discussed later when we examining the results from the system with the second set of material properties.

2.3.2 Influences of different parameters

The influences of different parameters on the stress fields arising in the single fiber pullout problem were studied in this section. The parameters considered here are fiber embedded length, the fiber/matrix diameter ratio, and the ratio between the Young’s moduli of fiber and matrix.
In the second set of material properties, we let the elastic modulus of the matrix, \( E_m \), to be 28.8 GPa, which is in the range of the elastic moduli for cement. Usually the elastic modulus for cement \( E_m \) is 10 – 45 GPa. The Poisson’s ratio, \( \nu_m \), is 0.2 for the matrix. So the bulk modulus becomes \( \kappa_m = E_m/(3 \times (1 - 2\nu_m)) = 16 \) GPa and the shear modulus is \( G_m = E_m/(2 \times (1 + \nu_m)) = 12 \) GPa.

Consider the reinforcement, i.e., the fiber, has the elastic modulus, \( E_f \), such that \( E_f/E_m = r_e \) and has the same Poisson’s ratio as the matrix, \( \nu_f = \nu_m = 0.2 \). Thus for different elastic constant ratio, \( r_e \), we have the bulk modulus of the fiber is \( \kappa_f = r_e \times \kappa_m \) and the shear modulus \( G_f = r_e \times G_m \). The fiber/matrix elastic constant ratio, \( r_e \) considered in this study is from 5 to 50. Although we can let the ratio go even higher, in reality it is beyond the materials we are interested in, i.e., cement based fiber reinforced materials. Further more, since we are interested in the influence of the chosen parameters, the calculation and the results have been nondimensionalized and normalized.

By using the finite element method, we can get the elastic stress field arising in the single fiber pullout test with different elastic constant ratio, fiber embedded length, and fiber/matrix diameter ratio. The overall trend of the stress field is similar to that described before. The stress concentration will occur at both fiber entry and embedded end. It is obtained that the level of stress concentration will be strongly influenced by the chosen parameters as discussed below.

**Effects of elastic constant ratio**

In order to see the effects of the fiber/matrix elastic constant ratio, we fixed the fiber embedded length and the fiber/matrix diameter ratio. The results for the axial stresses are shown in Figure 2.8-2.11. The axial stresses decrease rapidly near the fiber entry for the lower elastic constant ratio cases and rather uniformly for higher elastic constant ratio cases. Transfer of the fiber force into the matrix is characterized by a decay of the axial fiber
stress. In general, the high concentrations of the axial stresses arising near the fiber entry as well as at the fiber embedded end may result in a failure of the interphase, in particular since additional shear, radial and hoop stresses are active. It is obtained that the stress concentration level at the fiber entry decreases as the elastic constant ratio increases. In contrast, the stress concentration level at the fiber embedded end increases as the elastic constant ratio increases. Also the axial stresses in the middle part of the fiber, i.e., outside the region influenced by the singularity, increase as the elastic constant ratio increases. This reveals that for higher elastic constant ratio, more stresses remain in the fiber and less stress transferred into the matrix through the fiber/matrix shear stress implying higher pullout stress is needed to initiate the fiber/matrix interface debonding. In certain cases, the axial stresses at the fiber embedded end is not so small in comparing with the applied stresses. Thus the stress transfer through the fiber end must be taken into account if the embedded fiber end is not set to be free.

Similar behavior is also found for the shear and radial stresses, Figure 2.11-2.16. As the elastic constant ratio increases, both radial stress and shear stress will decrease at the fiber entry and increase at the fiber embedded end. The radial stresses are almost vanishing in the middle part of the interface, i.e., outside the region influenced by the singularities for all cases. It can be obtained from Figure 2.14-2.16 that the shear stress strongly decreases from the fiber entry towards the middle part of the fiber. At the region near the fiber embedded end the shear stress will increase again and reach a second maximum which is much lower than that at fiber entry. In other words, the load transfer through the interfacial shear stresses along the interface is very inhomogeneous. However, a more homogeneous situation can be found in the cases with higher elastic constant ratio. For which, the stress concentration level at the fiber entry is less pronounced and is enhanced at the fiber embedded end. Also the stresses in the middle part of the interface. If we consider the failure is controlled by the shear stresses, this more homogeneous behavior implies a higher applied force is needed for the initiation of the interface debonding. This result is consistent with previous discussion.
Effect of the fiber embedded length

By fixing the elastic constant ratio and the fiber/matrix diameter ratio, we can examine the effects of fiber embedded length. The results is shown in Figure 2.17-2.28. It is found that the fiber embedded length has almost no effect on the stress concentration level at the fiber entry for all cases examined here. For the case with low elastic constant ratio and fiber/matrix diameter ratio, the fiber embedded length does have slight influence on the stress concentration level at the fiber embedded end. The stress intensity at the fiber embedded end will become smaller and smaller as the fiber embedded length getting longer and longer. This is easy to explain. The longer embedded fiber means more interface to transfer the load from fiber into matrix. Since the stress intensity will not change at the fiber entry and the longer interface transferred more load into the matrix, it is nature that less stress remaining at the fiber embedded end. For the case with high fiber/matrix diameter ratio, this effect becomes less pronounced since the effects of the elastic constant ratio and the fiber/matrix diameter ratio are much more significant. It is also obtained that the trend of stress distribution does not change while the fiber embedded length changes except for the longer embedded fiber there is a bigger portion where the stress distributed uniformly. The effect of the fiber/matrix diameter ratio will be discussed later. The other interesting aspect is that when the fiber embedded length is long enough, the stress concentration level at the fiber embedded end will tend to converge to a certain value, see Figure 2.20 and 2.21. This follows the famous Saint-Venant's principle.

Effect of the fiber/matrix diameter ratio

The effects of fiber/matrix diameter ratio are also studied. Many researcher took this parameter as the approximate fiber volume fraction. This is true only for the aligned long fiber composite but not for the random short fiber composite. However, one should bear in mind that this ratio is still somewhat related to the fiber volume fraction. Similar to the effects of the elastic constant ratio, the stress concentration level at the fiber entry decreases
as the ratio increases while it increases at the fiber embedded end. However the effects of the fiber/matrix diameter ratio are less pronounced than those of the the elastic constant ratio. For the case with low elastic constant ratio, it is obtained that the axial stress at the middle part of the interface is almost constant Figure 2.29. This reveals, again, the Saint-Venant’s principle. The constant axial stress means that there is no load transfer from fiber into matrix. This can be seen in Figure 2.30. In which the shear stresses are zero at the middle part of the interface while the axial stresses are constant. It is also obtained that the higher fiber/matrix diameter ratio is the sooner the axial stress reaches constant. For the case with high elastic constant ratio, Figure 2.32-2.34, it can be found that the increase of fiber/matrix diameter ratio will improve the homogeneity of the stress distribution.

Effect of Poisson’s Ratio

The effects of Poisson’s ratio are also studied. We fixed the Poisson’s ratio of the matrix and then change that of the fiber. The main effects of Poisson’s ratio is on the radial stress and the shear stress. But it is rather small in comparison with the effects of elastic constant ratio and fiber/matrix diameter ratio. So it is reasonable not to consider the effect of Poisson’s ratio on the distribution of the elastic stress field. The results are shown in Figure 2.35-2.40.
2.4 Interpretation of interface debonding process

The single fiber pullout test is accomplished with the interface debonding followed by the fiber sliding. No matter what kind of debonding criterion is employed, it is directly related to the stress state at the interface, especially the stress concentration near the fiber entry and embedded end. Thus the understanding of the elastic stress fields arising in the interface can help us to have a better insight of interface debonding. Furthermore, we can have some basic idea about the limitation of existing models, e.g., the shear lag model, used for evaluating the experimental data.

In the case with high elastic constant ratio, we have obtained that a relatively large amount of axial stress remain in the fiber and the load transfer through the fiber bottom must be taken into account. It means that the interface at the bottom of fiber embedded end will subject to a high tensile stress. It is reasonable to expect the initiation of a mode I fracture at the bottom interface. This phenomenon had been observed in experimental studies, Betz(1982). The other concern is that the failure may occur in the matrix. If the single fiber pullout test was not properly designed, the specimen may fail before the interface debond thus no useful data can be obtained.

The stress concentration is directly associated with the interface debonding if the bond strength be taken as the interface debonding criterion. The peak values of stresses for different cases were shown in Figure 2.41-2.43. From Figure 2.42 we obtained that at certain cases the value of shear stresses at fiber embedded end will exceed the value at the fiber entry. In that case, it is reasonable to expect a mode II fracture to initiate at the fiber embedded end. It is also sown in Figure 2.43 that the absolute value of the radial stress at fiber embedded end in certain cases. But one should also notice that the radial stresses at the fiber embedded end are always negative. This means that the interface near the fiber embedded end is subjected to a compressive stress which can not induce a mode I fracture.
Subsequently one can only expect a mode I fracture from the fiber entry. However, it is mentioned before that the possibility of two-way debonding had been proposed by several researchers. Based on this argument, it is more reasonable to assume that the interface debonding is dominated by mode II fracture when bond strength was used as debonding criterion in the analysis.

2.5 Conclusion

The stress fields arising in the single fiber pullout problem have been studied by means of finite element analysis. The influences on the stress distribution along the fiber-matrix interface were also studied. It is obtained that the stress distribution is strongly influenced by the ratio of fiber and matrix elastic constants and the ratio of fiber/matrix diameter. It is also found that the effect of Poisson's ratio is relatively small comparing to other parameters and can be neglected. Based on the bond strength criterion, the initiation of interface debonding is dominated by mode II fracture. This supports many researchers' assumption. However it is also pointed out in this study that the initiation of a mode I fracture at the bottom of fiber embedded end may not be neglected in certain cases.

2.6 References


Fig. 2.1 Typical experimental load versus displacement curve (Maage 1978)
Fig. 2.2 Illustration of single fiber pull out test
Fig. 2.3 Idealized 2-D finite element model
Fig. 2.4.1 Axial stress distribution along the fiber/matrix interface

\[ L_m = 25 \; ; \; L/L_m = 60\% \; ; \; E_f = 73.5 \text{GPa} \; ; \; E_m = 2.3 \text{GPa} \; ; \; \nu_f = 0.18 \; ; \; \nu_m = 0.38 \]

- \( r = 0.0167 \) (fiber)
- \( r = 0.25 \) (fiber)
- \( r = 0.4833 \) (fiber)
- \( r = 0.5167 \) (matrix)
- \( r = 0.9833 \) (matrix)
Fig. 2.4.2 Axial stress distribution near the fiber entry

$L_m = 25$; $L/L_m = 60\%$; $E_f = 73.5\text{Gpa}$; $E_m = 2.3\text{Gpa}$; $v_f = 0.18$; $v_m = 0.38$
Fig. 2.4.3 Axial stress distribution near the fiber embedded end

$L_m = 25; L/L_m = 60\%; E_f = 73.5 \text{GPa}; E_m = 2.3 \text{GPa}; \nu_f = 0.18; \nu_m = 0.38$

- $r = 0.0167$ (fiber)
- $r = 0.25$ (fiber)
- $r = 0.4833$ (fiber)
- $r = 0.5167$ (matrix)
- $r = 0.9833$ (matrix)
Fig. 2.5.1  Radial stress distribution along the fiber/matrix interface

\[ L_m = 25; \quad L/L_m = 60\%; \quad E_1 = 73.5 \text{ Gpa}; \quad E_m = 2.3 \text{ Gpa}; \quad \nu_f = 0.18; \quad \nu_m = 0.38 \]

- ○ \( r = 0.0167 \) (fiber)
- □ \( r = 0.2500 \) (fiber)
- ⬤ \( r = 0.4833 \) (fiber)
- ○ ○ \( r = 0.5167 \) (matrix)
- ▲ \( r = 0.9833 \) (matrix)
Fig. 2.5.2  Radial stress distribution near the fiber entry

$L_m = 25; \frac{L}{L_m} = 60\%; E_t = 73.5 \text{ Gpa}; E_m = 2.3 \text{ Gpa}; v_t = 0.18; v_m = 0.38$

![Graph showing radial stress distribution near the fiber entry with various markers and lines representing different radial distances and material properties.](image-url)
Fig. 2.5.3  Radial stress distribution near the fiber embedded end

$L_m = 25$; $L/L_m = 60\%$; $E_f = 73.5$ GPa; $E_m = 2.3$ GPa; $\nu_f = 0.18$; $\nu_m = 0.38$

- $r = 0.0167$ (fiber)
- $r = 0.2500$ (fiber)
- $r = 0.4833$ (fiber)
- $r = 0.5167$ (fiber)
- $r = 0.9833$ (matrix)
- $\Delta r = 0.9833$ (matrix)
Fig. 2.6  Hoop stress distribution along the fiber/matrix interface

$L_n = 25 ; \frac{L_f}{L_m} = 60\% ; E_f = 73.5 \text{ Gpa} ; E_m = 2.3 \text{ Gpa} ; \nu_f = 0.18 ; \nu_m = 0.38$
Fig. 2.7.1 Shear stress distribution along the fiber/matrix interface

\[ L_m = 25; L/L_m = 60\% ; E_f = 73.5 \text{ Gpa} ; E_m = 2.3 \text{ Gpa} ; v_f = 0.18 ; v_m = 0.38 \]
Fig. 2.7.2 Shear stress distribution near the fiber entry

$L_m = 25; L/L_m = 60\%; E_f = 73.5 \text{ Gpa}; E_m = 2.3 \text{ Gpa}; \nu_f = 0.18; \nu_m = 0.38$

Longitude coordinate

Shear stress

-0.05
-0.10
-0.15

24.0
25.0
26.0

$r = 0.0167$ (fiber)
$r = 0.2500$ (fiber)
$r = 0.4833$ (fiber)
$r = 0.5167$ (matrix)
$r = 0.9833$ (matrix)
Fig. 2.7.3 Shear stress distribution near the fiber embedded end

\( L_m = 25 \); \( L/L_m = 60\% \); \( E_t = 73.5 \text{ Gpa} \); \( E_m = 2.3 \text{ Gpa} \); \( v_t = 0.18 \); \( v_m = 0.38 \)

Shear stress

Longitude coordinate
Fig. 2.8  \( \frac{R_i}{R_m} = 5\% ; \frac{L_i}{L_m} = 80\% ; L_m = 25 \) 

Axial stress distribution

- \( \frac{E_i}{E_m} = 5 \)
- \( \frac{E_i}{E_m} = 10 \)
- \( \frac{E_i}{E_m} = 20 \)
- \( \frac{E_i}{E_m} = 50 \)
- \( \frac{E_i}{E_m} = 100 \)
Fig. 2.9  \( L_m = 25 \); \( L/L_m = 40\% \); \( R/R_m = 10\% \)

Axial stress distribution

Axial stress

Longitude coordinate
Fig. 2.10  \( R/R_m = 20\% ; L/L_m = 80\% ; L_m = 25 \)

Axial stress distribution

- \( E/E_m = 5 \)
- \( E/E_m = 10 \)
- \( E/E_m = 20 \)
- \( E/E_m = 50 \)
- \( E/E_m = 100 \)
Fig. 2.11  \( \frac{R}{R_m} = 5\% \); \( \frac{L_r}{L_m} = 80\% \); \( L_m = 25 \)

Radial stress distribution
Fig. 2.12  \( L_m = 25 \); \( L/L_m = 40\% \); \( R/R_m = 10\% \)

Radial stress distribution

- \( E/E_m = 5 \)
- \( E/E_m = 10 \)
- \( E/E_m = 20 \)
- \( E/E_m = 50 \)
- \( E/E_m = 100 \)

Radial stress

Longitude coordinate
Fig. 2.13  $R/R_m = 20\%$ ; $L/L_m = 80\%$ ; $L_m = 25$

Radial stress distribution

Radial stress

Longitude coordinate
Fig. 2.14  $R/R_m = 5\%$ ; $L/L_m = 80\%$ ; $L_m = 25$
Shear stress distribution

Longitude coordinate

Shear stress

- $E/E_m = 5$
- $E/E_m = 10$
- $E/E_m = 20$
- $E/E_m = 50$
- $E/E_m = 100$
Fig. 2.15  \( L_m = 25 \); \( L/L_m = 40\% \); \( R/R_m = 10\% \)

Shear stress distribution

Shear stress

\[ 0.05 \quad 0.00 \quad -0.05 \quad -0.10 \quad -0.15 \quad -0.20 \quad -0.25 \quad -0.30 \quad -0.35 \quad -0.40 \]

Longitude coordinate

\[ 0.0 \quad 5.0 \quad 10.0 \quad 15.0 \quad 20.0 \quad 25.0 \]

- \( E_f/E_m = 5 \)
- \( E_f/E_m = 10 \)
- \( E_f/E_m = 20 \)
- \( E_f/E_m = 50 \)
- \( E_f/E_m = 100 \)
Fig. 2.16  \( r/r_m = 20\% \); \( L/L_m = 80\% \); \( L_m = 25 \)

Shear stress distribution

Shear stress

Longitude coordinate

- Shear stress distribution for different values of \( E_t/E_m \):
  - \( E_t/E_m = 5 \)
  - \( E_t/E_m = 10 \)
  - \( E_t/E_m = 20 \)
  - \( E_t/E_m = 50 \)
  - \( E_t/E_m = 100 \)
Fig. 2.17 $L_m = 25$; $R/R_m = 5\%$; $E/E_m = 10$

Axial stress distribution

- $L/L_m = 40\%$
- $L/L_m = 50\%$
- $L/L_m = 60\%$
- $L/L_m = 70\%$
- $L/L_m = 80\%$

Axial stress

Longitude coordinate
Fig. 2.18  $L_m = 25$ ; $R/R_m = 5\%$ ; $E/E_m = 10$

Shear stress distribution

Shear stress

Longitude coordinate

- $L/L_m = 40\%$
- $L/L_m = 50\%$
- $L/L_m = 60\%$
- $L/L_m = 70\%$
- $L/L_m = 80\%$
Fig. 2.19 $L_m = 25$; $R/R_m = 5\%$; $E/E_m = 10$

Radial stress distribution

- $L/L_m = 40\%$
- $L/L_m = 50\%$
- $L/L_m = 60\%$
- $L/L_m = 70\%$
- $L/L_m = 80\%$

Radial stress

Longitude coordinate
Fig. 2.20  \( L_m = 50 \); \( R_f/R_m = 5\% \); \( E_f/E_m = 10 \)

Axial stress distribution

- \( L_f/L_m = 40\% \)
- \( L_f/L_m = 50\% \)
- \( L_f/L_m = 60\% \)
- \( L_f/L_m = 70\% \)
- \( L_f/L_m = 80\% \)
Fig. 2.21 \( L_m = 50 ; R/R_m = 5\% ; E/E_m = 10 \)

Shear stress distribution

\[ \text{Shear stress} \]

\[ \text{Longitude coordinate} \]
Fig. 2.22  \( L_m = 50 \); \( R_f/R_m = 5\% \); \( E_f/E_m = 10 \)
Radial stress distribution

- \( L_f/L_m = 40\% \)
- \( L_f/L_m = 50\% \)
- \( L_f/L_m = 60\% \)
- \( L_f/L_m = 70\% \)
- \( L_f/L_m = 80\% \)

Radial stress

Longitude coordinate
Fig. 2.23  $L_m = 25$ ; $R/R_m = 5\%$ ; $E/E_m = 50$

Axial stress distribution

- $L/L_m = 40\%$
- $L/L_m = 50\%$
- $L/L_m = 60\%$
- $L/L_m = 70\%$
- $L/L_m = 80\%$

Axial stress vs Longitude coordinate
Fig. 2.24  $L_m = 25$; $R/R_m = 5\%$; $E/E_m = 50$

Shear stress distribution

- $L/L_m = 40\%$
- $L/L_m = 50\%$
- $L/L_m = 60\%$
- $L/L_m = 70\%$
- $L/L_m = 80\%$

Longitude coordinate

Shear stress

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Fig. 2.25  \( L_m = 25 ; R/R_m = 5\% ; E/E_m = 50 \)

Radial stress distribution

- \( L/L_m = 40\% \)
- \( L/L_m = 50\% \)
- \( L/L_m = 60\% \)
- \( L/L_m = 70\% \)
- \( L/L_m = 80\% \)

Longitude coordinate

Radial stress
Fig. 2.26  $L_m = 25$ ; $R/R_m = 20\%$ ; $E/E_m = 10$

Axial stress distribution

- $L/L_m = 40\%$
- $L/L_m = 50\%$
- $L/L_m = 60\%$
- $L/L_m = 70\%$
- $L/L_m = 80\%$
Fig. 2.27  $L_m = 25 ; \frac{R_f}{R_m} = 20\% ; \frac{E_f}{E_m} = 10$

Shear stress distribution
Fig. 2.28  \( L_m = 25 \); \( R/R_m = 20\% \); \( E/E_m = 10 \)

Radial stress distribution
Fig. 2.29  \( L_m = 25 \); \( L/L_m = 80\% \); \( E/E_m = 5 \)

Axial stress distribution

- \( r/r_m = 33\% \)
- \( r/r_m = 20\% \)
- \( r/r_m = 10\% \)
- \( r/r_m = 5\% \)
Fig. 2.30  \( L_m = 25 \); \( L/L_m = 80\% \); \( E/E_m = 5 \)

Shear stress distribution
Fig. 2.31  \( L_m = 25 ; \frac{L_f}{L_m} = 80\% ; \frac{E_f}{E_m} = 5 \)
Radial stress distribution

- \( r/r_m = 20\% \)
- \( r/r_m = 10\% \)
- \( r/r_m = 5\% \)
- \( r/r_{asm} = 33\% \)

Radial stress

Longitude coordinate
Fig. 2.32  \( L_m = 25 \); \( L/L_m = 80\% \); \( E_r/E_m = 50 \)

Axial stress distribution

- \( r/r_m = 20\% \)
- \( r/r_m = 10\% \)
- \( r/r_m = 5\% \)
- \( r/r_m = 2\% \)
Fig. 2.33  \( L_m = 25 \); \( L/L_m = 80\% \); \( E_r/E_m = 50 \)

Shear stress distribution

Shear stress vs Longitude coordinate
Fig. 2.34  \( L_m = 25 \); \( L/L_m = 80\% \); \( E_t/E_m = 50 \)
Radial stress distribution

\[ r/r_m = \{20\%, 10\%, 5\%, 2\%\} \]

Longitude coordinate

Radial stress
Fig. 2.35 Effects of Poisson’s ratio

\( \frac{E_f}{E_m} = 10; \, L_m = 25; \, \frac{L_f}{L_m} = 80\%; \, v_m = 0.2 \)

- \( v_f = 0.1 \)
- \( v_f = 0.2 \)
- \( v_f = 0.3 \)
- \( v_f = 0.4 \)

Axial stress

Longitude coordinate
Fig. 2.36  Effects of Poisson’s ratio
\[ \frac{E_f}{E_m} = 10; \quad L_m = 25; \quad \frac{L_f}{L_m} = 80\%; \quad \nu_m = 0.2 \]
Fig. 2.37 Effects of Poisson's ratio

\( \frac{E_r}{E_m} = 10; L_m = 25; L_r/L_m = 80\%; \nu_m = 0.2 \)

Radial stress

Longitude coordinate
Fig. 2.38 Effects of Poisson's ratio

\[ \frac{E_i}{E_m} = 50; \quad L_m = 25; \quad \frac{L_i}{L_m} = 80\%; \quad \nu_m = 0.2 \]
Fig. 2.38 Effects of Poisson's ratio

\( \frac{E_f}{E_m} = 50 \); \( L_m = 25 \); \( \frac{L_f}{L_m} = 80\% \); \( \nu_m = 0.2 \)
Fig. 2.39 Effects of Poisson’s ratio

$E_l/E_m = 50$; $L_m = 25$; $L_l/L_m = 80\%$; $\nu_m = 0.2$

---

Shear stress

Longitude coordinate

- $\nu_l = 0.1$
- $\nu_l = 0.2$
- $\nu_l = 0.3$
- $\nu_l = 0.4$
Fig. 2.40  Effects of Poisson’s ratio
\[
E_f/E_m = 50; \quad L_m = 25; \quad L_f/L_m = 80\%; \quad \nu_m = 0.2
\]

Radial stress

Longitude coordinate
Fig. 2.41 Peak values for the axial stresses
Fig. 2.42 Peak values for the shear stresses
Fig. 2.43 Peak values for the radial stresses
Chapter 3

Elastic Stress Fields Arising in the
Single Fiber Pullout System:
Analytical Solution

3.1 Introduction

Recent studies of fiber reinforced materials have shown that the matrix materials can be toughened by the fiber reinforcements through the fiber bridging and fiber/matrix interface debonding process. Thus the toughen mechanism of the fiber reinforced composites is mainly controlled by the fiber/matrix interface properties. The single fiber pull out test is one of the most popular mechanical test technique designed to determine the fiber/matrix interface properties. A significant effort has been put into the characterization of the interface properties through the experimental data, Lawrence(1972), Gopalaratnam and Shah(1987), Hsueh(1990). Most of the works done were based on the shear-lag model which has been found can not give an appropriate description of the stress field arising in the neighborhood of the fiber. The current work is trying to find a more rigorous analytical solution for the elastic stress field arising in the single fiber pullout problem. The problem is similar to the classical problem of the diffusion of load from a tension bar embedded in a three dimensional elastic medium. The problem is not new and various approaches have been published.
over the years (Muki and Sternberg, 1969, 1970; Luk and Keer, 1979; Whitney and Drzal, 1987; Hsueh, 1988; Shetty, 1988; Gerhardt and Cheng, 1989) These works involve different types and levels of approximations such as: modeling the fiber as rigid, shear lag analysis, and energy method. Kurtz and Pagano (1991) proposed a three dimensional elasticity infinite series solution to the fiber pullout problem within the confines of elasticity. However, the problem they considered is only a special case of the fiber pullout type problem. The objective of this work is to solve a more general problem by modifying Kurtz and Pagano's frame work. The motivation behind this effort is to establish a baseline which the finite element results can compare to. By studying the elastic stress fields, one can get some general ideas about the effective coefficients on the stress fields arising in the single fiber pull-out system. Both fiber and matrix considered in current work are linear and isotropic. The interface between fiber and matrix is assumed to be perfectly bonded. Certain mathematical difficulties were encountered and will be discussed in the following sections.
3.2 Theory and Formulation

3.2.1 Single fiber pullout model

Consider a single fiber pullout model consisting of single fiber partially embedded in a cylindrical matrix as shown in Figure 3.1. The bottom of the matrix is fixed and the free end of the fiber is subjected to a pull load $\pi a^2 \sigma_0$. Both the fiber and matrix are linear elastic and isotropic materials. The fiber cross section is assumed to be a circle with radius $a$ and the radius of the matrix is $b$. The center of the fiber is coincide with the matrix’s center. According to the axisymmetric property of the idealized model, the 3-D problem can be reduced to a 2-D problem. The mathematics of the problem are thus be significantly simplified. The interface between fiber and matrix is assumed to be perfectly bonded. The primary interest in this study is to get an analytical solution of the elastic stress field of the interface especially in the neighborhood of the fiber entry and fiber embedded end.

The 2-D model is divided into three parts, one for fiber and two for matrix, as shown in Figure 3.2. In order to distinguish between these three components, we will designate the fiber by superscript $f$ and the matrix material by superscripts $m_1$ and $m_2$. The associated boundary conditions for this problem can be broken down into three groups: interface continuity conditions, free surface boundary conditions, and geometry boundary conditions. These boundary conditions are stated as:

*interface continuity*

\[
\begin{align*}
    u_z^{m_1}(a, z) &= u_z^f(a, z) & l & \leq z \leq L \\
    u_r^{m_1}(a, z) &= u_r^f(a, z) & l & \leq z \leq L \\
    u_z^{m_1}(a, z) &= u_z^{m_2}(a, z) & 0 & \leq z \leq l \\
    u_r^{m_1}(a, z) &= u_r^{m_2}(a, z) & 0 & \leq z \leq l 
\end{align*}
\]
\[ \sigma_{r z}^{m1}(a, z) = \sigma_{r z}^{f1}(a, z) \quad l \leq z \leq L \quad (3.5) \]
\[ \tau_{rz}^{m1}(a, z) = \tau_{rz}^{f1}(a, z) \quad l \leq z \leq L \quad (3.6) \]
\[ \sigma_{r z}^{m1}(a, z) = \sigma_{r z}^{m2}(a, z) \quad 0 \leq z \leq l \quad (3.7) \]
\[ \sigma_{z z}^{m1}(a, z) = \sigma_{z z}^{m2}(a, z) \quad 0 \leq z \leq l \quad (3.8) \]
\[ \tau_{rz}^{m1}(a, z) = \tau_{rz}^{m2}(a, z) \quad 0 \leq z \leq l \quad (3.9) \]
\[ u_{x}^{m2}(r, l) = u_{x}^{f2}(r, l) \quad 0 \leq r \leq a \quad (3.10) \]
\[ u_{r}^{m2}(r, l) = u_{r}^{f2}(r, l) \quad 0 \leq r \leq a \quad (3.11) \]
\[ \sigma_{x}^{m2}(r, l) = \sigma_{x}^{f2}(r, l) \quad 0 \leq r \leq a \quad (3.12) \]
\[ \tau_{rz}^{m2}(r, l) = \tau_{rz}^{f2}(r, l) \quad 0 \leq r \leq a \quad (3.13) \]

**Free surface boundary condition**

\[ \sigma_{r z}^{m1}(b, z) = 0 \quad 0 \leq z \leq L \quad (3.14) \]
\[ \tau_{rz}^{m1}(b, z) = 0 \quad 0 \leq z \leq L \quad (3.15) \]
\[ \tau_{r z}^{f1}(r, L) = 0 \quad 0 \leq r \leq a \quad (3.16) \]
\[ \sigma_{r}^{f}(r, L) = \sigma_{0} \quad 0 \leq r \leq a \quad (3.17) \]
\[ \sigma_{x}^{m1}(r, L) = 0 \quad a \leq r \leq b \quad (3.18) \]
\[ \tau_{rz}^{m1}(r, L) = 0 \quad a \leq r \leq b \quad (3.19) \]

**Fixed end boundary condition**

\[ u_{z}^{m1}(r, 0) = 0 \quad a \leq r \leq b \quad (3.20) \]
\[ u_{z}^{m2}(r, 0) = 0 \quad 0 \leq r \leq a \quad (3.21) \]
3.2.2 Axisymmetric boundary value problem

Since the boundary value problem is axisymmetric, the stress components are independent of the coordinate $\theta$. It follows that all the derivatives with respect to $\theta$ vanish. Also, the shear stresses acting in the plane containing the axis of revolution, i.e., $\tau_{r\theta}$ and $\tau_{z\theta}$, vanish due to symmetry. The equilibrium equations for both fiber and matrix are then reduced to

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$  \hspace{1cm} (3.22)

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0$$  \hspace{1cm} (3.23)

The engineering strain components remaining in the axisymmetric problem are:

$$\varepsilon_r = \frac{\partial U_r}{\partial r}, \quad \varepsilon_\theta = \frac{U_r}{r}, \quad \varepsilon_z = \frac{\partial U_z}{\partial z}, \quad \gamma_{rz} = \frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r}. \hspace{1cm} (3.24)$$

And the Hook’s Law can be expressed as:

$$\sigma_i = C_{ij} \varepsilon_j \quad i, j = 1, 2, 3, 6$$  \hspace{1cm} (3.25)

where the notation 1, 2, 3 and 6 are corresponding to $z$, $r$, $\theta$, and $rz$ respectively.

Equations (3.22)-(3.25) represent the complete system of governing equations for the problem. Since both fiber and matrix are assumed to be isotropic materials, we can simplify the problem by using Love’s stress function (Love, 1944). The field equations for each material are then reduced to a single differential equation:

$$\nabla^4 \phi = 0$$  \hspace{1cm} (3.26)

where

$$\nabla^4 = \left\{ \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} \right\}^2.$$

The stress and displacement fields can then be derived from:

$$\sigma_r = \frac{\partial}{\partial z} \left[ \nu \nabla^2 \phi - \frac{\partial^2 \phi}{\partial r^2} \right]$$  \hspace{1cm} (3.27)
\[
\sigma_\theta = \frac{\partial}{\partial z} \left( \nu \nabla^2 \phi - \frac{\partial \phi}{r \partial r} \right) \quad (3.28)
\]
\[
\sigma_z = \frac{\partial}{\partial z} \left[ (2 - \nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right] \quad (3.29)
\]
\[
\tau_{rz} = \frac{\partial}{\partial r} \left[ (1 - \nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right] \quad (3.30)
\]
\[
2GU_r = -\frac{\partial^2 \phi}{\partial r \partial z} \quad (3.31)
\]
\[
2GU_z = 2(1 - \nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \quad (3.32)
\]

where \( \nu \) and \( G \) denote Poisson's ratio and the shear modulus respectively.

It should be noticed that the current model was divided into three components and equation (3.26) has to be solved independently for each component. However, the superscriptions used to denote those three components, i.e., \( f \), \( m1 \), and \( m2 \), were omitted in eqns (3.22)-(3.32) since these equations are identical for all three components except the superscription.

Several solutions for cylindrical problems considering only one constituent material can be found in the literature such as Barton and Ithaca 1941; Rankin and Schenectady, 1944; Little and Childs, 1967; Power and Childs, 1971; Moghe and Neff, 1971; Benthem and Minderhoud, 1972. These solutions provided a guideline in choosing stress functions for both fiber and matrix in Kurtz and Pagano's previous work. For simplicity, the current study employed the stress functions used in Pagano's work and made some modifications. The actual Love's stress functions used in current study to solve the boundary value problem are:

\[
\phi' = \sum_{i=1}^{\infty} \left[ A_{1i} I_0(k_i r) + A_{2i} k_i r I_1(k_i r) \right] \sin(k_i z) \\
+ \sum_{j=1}^{\infty} \left[ C_{1j} z \cosh(\alpha_j z) + C_{2j} \sinh(\alpha_j z) \right] e^{-\alpha_j z} J_0(\alpha_j r) + F_1 r^2 z + F_3 z^3
\]

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\[
\phi^{m1} = \sum_{i=1}^{\infty} \left[ B_{1i}K_0(\vec{k}_i r) + B_{2i}\vec{k}_i r K_1(\vec{k}_i r) + B_{3i}I_0(\vec{k}_i r) + B_{4i}\vec{k}_i r I_1(\vec{k}_i r) \right] \sin(\vec{k}_i z) \\
+ \sum_{j=1}^{\infty} \left[ D_{1j}z \cosh(\lambda_j z) + D_{2j}z \sinh(\lambda_j z) \right] e^{-\lambda_j L} \left[ J_0(\lambda_j r) + \mu_j Y_0(\lambda_j r) \right]
\]
\[
+ G_1 r^2 z + G_2 z \ln(r) + G_3 z^3
\]

\[
\phi^{m2} = \sum_{i=1}^{\infty} \left[ A_{3i}I_0(\vec{\bar{k}}_i r) + A_{4i}\vec{\bar{k}}_i r I_1(\vec{\bar{k}}_i r) \right] \sin(\vec{\bar{k}}_i z) \\
+ \sum_{j=1}^{\infty} \left[ C_{3j}z \cosh(\alpha_j z) + C_{4j}z \sinh(\alpha_j z) \right] e^{-\alpha_j L} J_0(\alpha_j r) + F_3 r^2 z + F_7 z^3
\]

where \(J\) and \(Y\) are Bessel functions of first and second kind, respectively. \(I\) and \(K\) are the corresponding modified Bessel functions.

It is interesting to point out that these stress functions are actually superposition of three well-known solutions. The first summation term in the stress function is similar to the solution for an infinite long fiber embedded in an infinite matrix. The second summation term is similar to the solution for a penny-shaped crack or an axisymmetric band of pressure applied to surfaces of an elastic body of finite thickness(Sneddon, 1946). The last portion of the stress function is the far-field or steady-state solution. The negative exponential power was multiplied to the hyperbolic sin and cosine terms in the stress functions to keep the functions bounded on the intervals considered.

The expressions for the stresses and displacements can be developed by substituting eqn (3.33)-(3.35) into eqns (3.27)-(3.32). These expressions, as shown in the Appendix, were further substituted into eqns (3.1)-(3.21) to form a system of linear equations. Boundary conditions (3.20) and (3.21) are identically satisfied by the choice of the stress functions. In order to use the remaining nineteen boundary conditions to develop a linear system with infinite set of equations and infinite unknowns, the orthogonality in each set of functions is needed. The orthogonality in each set of functions is achieved by properly choosing the value of constants \(k_i, \vec{k}_i, \vec{\bar{k}}_i, \alpha_j, \lambda_j, \) and \(\mu_j\).
The constants \( k_i, \bar{k}_i, \) and \( \tilde{k}_i \) are eigenvalues defined by the following relationships:

\[
\sin[k_i(L - l)] = 0 \quad i = 1, 2, 3 \ldots \quad (3.36)
\]
\[
\sin[\bar{k}_i L] = 0 \quad i = 1, 2, 3 \ldots \quad (3.37)
\]
\[
\sin[\tilde{k}_i l] = 0 \quad i = 1, 2, 3 \ldots \quad (3.38)
\]

making each set of functions to be orthogonal on the intervals \( l \leq z \leq L, \) \( 0 \leq z \leq L, \) and \( 0 \leq z \leq l, \) respectively. The constant \( \alpha_j \) is eigenvalue defined by the relationship:

\[
J_1(\alpha_j a) = 0 \quad j = 1, 2, 3 \ldots \quad (3.39)
\]

to make the set of functions orthogonal over the interval \( 0 \leq r \leq a. \) The constants \( \mu_j \) and \( \lambda_j \) are defined by the relationships:

\[
\mu_j = -\frac{J_1(\lambda_j a)}{Y_1(\lambda_j a)} \quad (3.40)
\]
\[
J_1(\lambda_j b) + \mu_j Y_1(\lambda_j b) = 0 \quad j = 1, 2, 3 \ldots \quad (3.41)
\]

which define a set of orthogonal function on the interval of \( a \leq r \leq b. \)

3.2.3 The linear system of equations

The linear system with infinite number of equations and infinite unknowns is then can be established by substituting the expressions for the stresses and displacements into eqns (BA)-(BS) and using the orthogonality of the functions. The following contractions are used in the expression for the linear system.

\[
I_0(k_i r), I_1(k_i r) \rightarrow I_0, I_1 \\
I_0(\bar{k}_i r), I_1(\bar{k}_i r) \rightarrow \bar{I}_0, \bar{I}_1 \\
I_0(\tilde{k}_i r), I_1(\tilde{k}_i r) \rightarrow \tilde{I}_0, \tilde{I}_1 \\
K_0(k_i r), K_1(k_i r) \rightarrow K_0, K_1
\]
Substituting the expressions for the displacements and stresses into eqns (3.1) and (3.6), multiplying by \( \sin(k_3) \) and then integrating with respect to \( z \) over the interval \( l \leq z \leq L \), yields the following equations:

\[
\begin{align*}
\sum_{k=1}^{\infty} \left[ B_{1k} k_1^2 \tilde{K}_0 + B_{2k} k_1^2 \left[ (-4 + 4\nu_m) \tilde{K}_0 + kr \tilde{K}_1 \right] \right] \int_{l}^{L} \sin(k_1 z) \sin(k_3 z) dz \\
+ \sum_{k=1}^{\infty} \left[ B_{3k} k_1^2 \tilde{I}_0 + B_{4k} k_1^2 \left[ (4 - 4\nu_m) \tilde{I}_0 + kr \tilde{I}_1 \right] \right] \int_{l}^{L} \sin(k_1 z) \sin(k_3 z) dz \\
+ \sum_{k=1}^{\infty} D_{1k} e^{-\lambda_k L} X_0 \int_{l}^{L} \left[ (2 - 4\nu_m) \lambda_k \sinh(\lambda_k z) - \lambda_k^2 z \cosh(\lambda_k z) \right] \sin(k_3 z) dz \\
- \sum_{k=1}^{\infty} D_{2k} e^{-\lambda_k L} X_0 \lambda_k^2 \int_{l}^{L} \sinh(\lambda_k z) \sin(k_3 z) dz + 8G_1 (1 - \nu_m) \int_{l}^{L} z \sin(k_3 z) dz \\
+ 6G_3 (1 - 2\nu_m) \int_{l}^{L} z \sin(k_3 z) dz - \gamma \left\{ A_{1j} k_j^2 I_0 + A_{2j} k_j^2 \left[ (4 - 4\nu_f) I_0 + kj I_1 \right] \right\} \frac{L - l}{2} \\
- \gamma \sum_{k=1}^{\infty} C_{1k} e^{-\alpha_k L} J_0 \left[ (2 - 4\nu_f) \alpha_k \sinh(\alpha_k z) - \alpha_k^2 z \cosh(\alpha_k z) \right] \sin(k_3 z) dz \\
+ \gamma \sum_{k=1}^{\infty} C_{2k} e^{-\alpha_k L} J_0 \alpha_k^2 \int_{l}^{L} \sinh(\alpha_k z) \sin(k_3 z) dz - \gamma 8F_1 (1 - \nu_f) \int_{l}^{L} z \sin(k_3 z) dz \\
- \gamma 6F_3 (1 - 2\nu_f) \int_{l}^{L} z \sin(k_3 z) dz = 0 \quad r = a; \; j = 1, 2, 3, \ldots \quad (3.42)
\end{align*}
\]
\[- \sum_{i=1}^{\infty} \{B_{1i} \bar{k}^3 \bar{K}_1 - B_{2i} \bar{k}^3 [(2 - 2\nu_m)\bar{K}_1 - \bar{k}r\bar{K}_0]\} \int_{l}^{L} \sin(\bar{k}_i z) \sin(\bar{k}_j z) dz \]

\[+ \sum_{i=1}^{\infty} \{B_{3i} \bar{k}^3 \bar{I}_1 + B_{4i} \bar{k}^3 [(2 - 2\nu_m)\bar{I}_1 + \bar{k}r\bar{I}_0]\} \int_{l}^{L} \sin(\bar{k}_i z) \sin(\bar{k}_j z) dz \quad (3.43)\]

\[- \{A_{1j} \bar{k}^3 \bar{I}_1 + A_{2j} \bar{k}^3 [(2 - 2\nu_f)\bar{I}_1 + k_j r\bar{I}_0]\} \frac{L-l}{2} = 0 \quad r = a; \quad j = 1, 2, 3 \ldots \]

Note that the orthogonality, \(\int_{l}^{L} \sin(\bar{k}_i z) \sin(\bar{k}_j z) dz = 0 \) if \( i \neq j \), is used to get equations (3.42) and (3.43). Similarly, multiplying \(\cos(\bar{k}_i z)\) to eqns (3.5) and (3.2) and then integrating from \(l\) to \(L\) with respect to \(z\), yields

\[- \sum_{i=1}^{\infty} \{B_{1i} \bar{k}^3 \bar{K}_0 + \frac{\bar{k}^2}{r} \bar{K}_1\} - B_{2i} \bar{k}^3 [(1 - 2\nu_m)\bar{K}_0 - \bar{k}r\bar{K}_1]\} \int_{l}^{L} \cos(\bar{k}_i z) \cos(\bar{k}_j z) dz \]

\[- \sum_{i=1}^{\infty} \{B_{3i} \bar{k}^3 \bar{I}_0 - \frac{\bar{k}^2}{r} \bar{I}_1 + B_{4i} \bar{k}^3 [(1 - 2\nu_m)\bar{I}_0 + \bar{k}r\bar{I}_1]\} \int_{l}^{L} \cos(\bar{k}_i z) \cos(\bar{k}_j z) dz \]

\[+ \sum_{k=1}^{\infty} D_{1k} e^{-\lambda_k L} \lambda_k^2 X_0 \int_{l}^{L} \{1 + 2 \nu_m\} \cosh(\lambda_k z) + \lambda_k z \sinh(\lambda_k z)\} \cos(\bar{k}_j z) dz \]

\[+ \sum_{k=1}^{\infty} D_{2k} \lambda_k e^{-\lambda_k L} \lambda_k^2 X_0 \int_{l}^{L} \cosh(\lambda_k z) \cos(\bar{k}_j z) dz \]

\[+ \{G_1(1 - 2 + 4\nu_m) + \frac{G_2}{r^2} + 6G_3 \nu_m + G_1(2 - 4\nu_f) - 6G_3 \nu_f\} \int_{l}^{L} \cos(\bar{k}_i z) dz \]

\[- \{A_{1j} \lambda_j [-\bar{k}^2 \bar{I}_0 + \frac{\bar{k}}{r} \bar{I}_1] - A_{2j} \bar{k}^3 [(1 - 2\nu_f)\bar{I}_0 + k_j r\bar{I}_1]\} \frac{L-l}{2} \]

\[+ \sum_{k=1}^{\infty} C_{1k} e^{-\alpha_k L} J_0 \alpha_k^2 \int_{l}^{L} \{(1 - 2\nu_f) \cosh(\alpha_k z) + \alpha_k z \sinh(\alpha_k z)\} \cos(\bar{k}_j z) dz \]

\[+ \sum_{k=1}^{\infty} C_{2k} e^{-\alpha_k L} \alpha_k^2 J_0 \int_{l}^{L} \alpha_k \cosh(\alpha_k z) \cos(\bar{k}_j z) dz = 0 \quad r = a; \quad j = 1, 2, 3 \ldots \]

\[(3.44)\]
\[
\sum_{i=1}^{\infty} \left\{ B_{1i} k^2 R_1 + B_{2i} k^3 r R_0 - B_{3i} k^2 I_1 - B_{4i} k^3 r I_0 \right\} \int_0^L \cos(k_i z) \cos(k_j z) \, dz \\
+ \gamma [A_{1j} k^2 I_1 + \gamma A_{2j} k^3 r I_0] \frac{L - l}{2} \\
- [2G_1 r + \frac{G_2}{r} - 2\gamma F \gamma r] \int_0^L \cos(k_j z) \, dz = 0 \quad r = a; \ j = 1, 2, 3.....
\] (3.45)

Multiplying eqns (3.4) and (3.3) by \( \cos(k_j z) \) and \( \sin(k_j z) \) respectively, integrating from 0 to \( l \) with respect to \( z \), and using the principle of orthogonality, yields

\[
\sum_{i=1}^{\infty} \left\{ B_{1i} k^2 R_1 + B_{2i} k^3 r R_0 - B_{3i} k^2 I_1 - B_{4i} k^3 r I_0 \right\} \int_0^l \cos(k_i z) \cos(k_j z) \, dz \\
+ [A_{3j} k^2 I_1 + A_{4j} k^3 r I_0] \frac{l}{2} = 0 \quad r = a; \ j = 1, 2, 3.....
\] (3.46)

\[
\sum_{i=1}^{\infty} \left\{ B_{1i} k^2 R_0 + B_{2i} [(-4 + 4\nu_m) k^2 R_0 + k^3 r R_1] \right\} \int_0^l \sin(k_i z) \sin(k_j z) \, dz \\
+ \sum_{i=1}^{\infty} \left\{ B_{3i} k^2 I_0 + B_{4i} [(4 - 4\nu_m) k^2 I_0 + k^3 r I_1] \right\} \int_0^l \sin(k_i z) \sin(k_j z) \, dz \\
+ \sum_{k=1}^{\infty} D_{1k} e^{-\lambda_k L} X_0 \int_0^l [(2 - 4\nu_m) \lambda_k \sinh(\lambda_k z) - \lambda_k^2 z \cosh(\lambda_k) \sin(k_i z) \cos(k_i z) \, dz \\
- \sum_{k=1}^{\infty} D_{2k} e^{-\lambda_k L} X_0 \lambda_k \int_0^l \sinh(\lambda_k z) \sin(k_i z) \, dz + 8G_1 (1 - \nu_m) \int_0^l z \sin(k_i z) \, dz \\
+ 6G_3 (1 - 2\nu_m) \int_0^l z \sin(k_i z) \, dz - \{ A_{3j} k^2 I_1 + A_{4j} k^3 r [(4 - 4\nu_m) I_0 + k^3 r I_1] \} \frac{l}{2} \\
- \sum_{k=1}^{\infty} C_{3k} e^{-\alpha_k L} J_0 \int_0^l [(2 - 4\nu_m) \alpha_k \sinh(\alpha_k z) - \alpha_k^2 z \cosh(\alpha_k) \sin(k_j z) \cos(k_j z) \, dz \\
+ \sum_{k=1}^{\infty} C_{4k} e^{-\alpha_k L} J_0 \alpha_k \int_0^l \sinh(\alpha_k z) \sin(k_j z) \, dz \\
- [8F_3 (1 - \nu_m) + 6F_7 (1 - 2\nu_m)] \int_0^l z \sin(k_j z) \, dz = 0 \quad r = a; \ j = 1, 2, 3.....
\] (3.47)
Integrating eqns (3.7), (3.8), and (3.9) over the interval \(0 \leq z \leq l\) with respect to \(z\), yields

\[
\begin{align*}
- \sum_{i=1}^{\infty} \left\{ B_{2i}\bar{k}_i^3 \left[ \bar{k}_i \bar{R}_0 + \frac{1}{r} \bar{R}_1 \right] - B_{2i}\bar{k}_i^3 \left[ (1 - 2\nu_m)\bar{R}_0 - \bar{k}_i r \bar{R}_1 \right] \right\} \int_{0}^{l} \cos (\bar{k}_i z) \, dz \\
- \sum_{i=1}^{\infty} \left\{ B_{3i}\bar{k}_i^3 \left[ \bar{k}_i \bar{I}_0 - \frac{1}{r} \bar{I}_1 \right] + B_{4i}\bar{k}_i^3 \left[ (1 - 2\nu_m)\bar{I}_0 + \bar{k}_i r \bar{I}_1 \right] \right\} \int_{0}^{l} \cos (\bar{k}_i z) \, dz \\
+ \sum_{i=1}^{\infty} D_{1i} e^{-\lambda_k L} \left\{ [(1 + 2\nu_m)\lambda_k^2 X_0] \int_{0}^{l} \cosh (\lambda_k z) \, dz + [\lambda_k^3 X_0] \int_{0}^{l} z \sinh (\lambda_k z) \, dz \right\}
+ \sum_{i=1}^{\infty} D_{2i} e^{-\lambda_k L} [\lambda_k^3 X_0] \int_{0}^{l} \cosh (\lambda_k z) \, dz - G_1 l (2 - 4\nu_m) + G_2 \frac{l}{r^2} + 6G_3 \nu_m \\
+ \sum_{i=1}^{\infty} C_{3i} e^{-\alpha_k L} \left\{ [(1 - 2\nu_m)\alpha_k^2 J_0] \int_{0}^{l} \cosh (\alpha_k z) \, dz + [\alpha_k^3 J_0] \int_{0}^{l} z \sinh (\alpha_k z) \, dz \right\}
+ \sum_{i=1}^{\infty} C_{4i} e^{-\alpha_k L} \alpha_k^3 J_0 \int_{0}^{l} \cosh (\alpha_k z) \, dz + F_5 l (2 - 4\nu_m) - 6F_7 l \nu_m = 0 \quad r = a
\end{align*}
\]

\[
\begin{align*}
\sum_{i=1}^{\infty} \left\{ B_{2i}\bar{k}_i^3 \left[ \bar{k}_i \bar{R}_0 - \frac{1}{r} \bar{R}_1 \right] - B_{2i}\bar{k}_i^3 \left[ (4 - 2\nu_m)\bar{R}_0 - \bar{k}_i r \bar{R}_1 \right] \right\} \int_{0}^{l} \cos (\bar{k}_i z) \, dz \\
+ \sum_{i=1}^{\infty} \left\{ B_{3i}\bar{k}_i^3 \bar{I}_0 + B_{4i}\bar{k}_i^3 \left[ (4 - 2\nu_m)\bar{I}_0 + \bar{k}_i r \bar{I}_1 \right] \right\} \int_{0}^{l} \cos (\bar{k}_i z) \, dz \\
+ \sum_{i=1}^{\infty} D_{1i} e^{-\lambda_k L} \lambda_k^2 X_0 \int_{0}^{l} [(1 - 2\nu_m) \cosh (\lambda_k z) - \lambda_k z \sinh (\lambda_k z)] \, dz \\
+ \sum_{i=1}^{\infty} D_{2i} e^{-\lambda_k L} \lambda_k^3 X_0 \int_{0}^{l} \cosh (\lambda_k z) \, dz + G_1 l (8 - 4\nu_m) + 6G_3 l (1 - \nu_m) \\
- \sum_{i=1}^{\infty} C_{3i} e^{-\alpha_k L} \alpha_k^2 J_0 \int_{0}^{l} [(1 - 2\nu_m) \cosh (\alpha_k z) - \alpha_k z \sinh (\alpha_k z)] \, dz \\
+ \sum_{i=1}^{\infty} C_{4i} e^{-\alpha_k L} \alpha_k^3 J_0 \int_{0}^{l} \cosh (\alpha_k z) \, dz \\
- F_5 l (8 - 4\nu_m) - 6F_7 l (1 - \nu_m) = 0 \quad r = a
\end{align*}
\]
\[- \sum_{i=1}^{\infty} \left\{ B_{1i} \bar{k}_i^3 \bar{K}_1 - B_{2i} \bar{k}_i^3 \left[ (2 - 2\nu_m) \bar{K}_1 - \bar{k}_i r \bar{K}_0 \right] \right\} \int_0^l \sin(\bar{k}_i z)dz \]
\[+ \sum_{i=1}^{\infty} \left\{ B_{3i} \bar{k}_i^3 \bar{I}_1 + B_{4i} \bar{k}_i^3 \left[ (2 - 2\nu_m) \bar{I}_1 + \bar{k}_i r \bar{I}_0 \right] \right\} \int_0^l \sin(\bar{k}_i z)dz \]
\[+ \sum_{i=1}^{\infty} \left\{ A_{3i} \bar{k}_i^3 \bar{I}_1 + A_{4i} \bar{k}_i^3 \left[ (2 - 2\nu_m) \bar{I}_1 + \bar{k}_i r \bar{I}_0 \right] \right\} \int_0^l \sin(\bar{k}_i z)dz = 0 \quad r = a \] (3.50)

Multiplying eqns (3.14) and (3.15) by \( \cos(\bar{k}_j z) \) and \( \sin(\bar{k}_j z) \) respectively, integrating from 0 to \( L \) with respect to \( z \), and using the principle of orthogonality, yields

\[- \sum_{i=1}^{\infty} \left\{ B_{1i} \bar{k}_i^3 \bar{K}_0 + \frac{1}{r} \bar{K}_1 \right\} L \int_0^L \cos^2(\bar{k}_i z)dz + B_{2i} \bar{k}_i^3 \left[ (1 - 2\nu_m) \bar{K}_0 - \bar{k}_i r \bar{K}_1 \right] \frac{L}{2} \]
\[- B_{3i} \bar{k}_i^3 \left[ \bar{I}_0 - \frac{1}{\bar{I}_1} \right] \frac{L}{2} + B_{4i} \bar{k}_i^3 \left[ (-1 + 2\nu_m) \bar{I}_0 - \bar{k}_i r \bar{I}_1 \right] \frac{L}{2} \]
\[+ \sum_{k=1}^{\infty} D_{1k} e^{-\lambda_k L} \left\{ \lambda_k^2 \bar{X}_0 \int_0^L \left[ \cosh(\lambda_k z) + \lambda_k z \sinh(\lambda_k z) \right] \cos(\bar{k}_i z)dz \right\} \]
\[+ 2\nu_m \lambda_k^2 \bar{X}_0 \int_0^L \cosh(\lambda_k z) \cos(\bar{k}_i z)dz \]
\[+ \sum_{k=1}^{\infty} D_{2k} e^{-\lambda_k L} \lambda_k^3 \bar{X}_0 \int_0^L \cosh(\lambda_k z) \cos(\bar{k}_i z)dz = 0 \quad r = b ; \ i = 1, 2, 3 \ldots \] (3.51)

Substituting the expressions for displacements into eqns (3.10) and (3.11), multiplying by \( r J_0(\alpha_j r) \) and \( r J_1(\alpha_j r) \) respectively, and integrating from 0 to \( a \) with respect to \( r \), yields

\[\left\{ -B_{1i} \bar{k}_i^3 \bar{K}_1 - B_{2i} \bar{k}_i^3 \left[ (2 - 2\nu_m) \bar{K}_1 - \bar{k}_i r \bar{K}_0 \right] \right\} \frac{L}{2} \]
\[+ \left\{ B_{3i} \bar{k}_i^3 \bar{I}_1 + B_{4i} \bar{k}_i^3 \left[ (2 - 2\nu_m) \bar{I}_1 + \bar{k}_i r \bar{I}_0 \right] \right\} \frac{L}{2} = 0 \quad r = b ; \ i = 1, 2, 3 \ldots \] (3.52)
\[ + C_{3j} e^{-\alpha_j L} [\alpha_j (2 - 4\nu_m) \sinh(\alpha_j z) - \alpha_j^2 z \cosh(\alpha_j z)] \int_0^a r J_0^2(\alpha_j r) dr \]

\[ - C_{4j} \alpha_j^2 \sinh(\alpha_j z) \int_0^a r J_0^2(\alpha_j r) dr \]

\[- \gamma \sum_{i=1}^{\infty} A_{1i} k_i^2 \sin(k_i z) \int_0^a r I_0(k_i r) J_0(\alpha_j r) dr \]

\[- \gamma \sum_{i=1}^{\infty} A_{2i} k_i^2 \sin(k_i z) \int_0^a [(4 - 4\nu_f) I_0(k_i r) + k_i r I_1(k_i r)] r J_0(\alpha_j r) dr \]

\[- \gamma C_{1j} e^{-\alpha_j L} [\alpha_j (2 - 4\nu_f) \sinh(\alpha_j z) - \alpha_j^2 z \cosh(\alpha_j z)] \int_0^a r J_0^2(\alpha_j r) dr \]

\[- \gamma C_{2j} e^{-\alpha_j L} \alpha_j^2 \sinh(\alpha_j z) \int_0^a r J_0^2(\alpha_j r) dr = 0 \quad z = l; \ j = 1, 2, 3, \ldots \] (3.53)

\[- \sum_{i=1}^{\infty} k_i^2 \cos(k_i z) \{ A_{3i} \int_0^a r \tilde{I}_1(k_i r) J_1(\alpha_j r) dr + A_{4i} k_i \int_0^a r^2 \tilde{I}_0(k_i r) J_1(\alpha_j r) dr \} \]

\[+ e^{-\alpha_j L} \{ C_{3j} [\alpha_j \cosh(\alpha_j z) + \alpha_j^2 z \sinh(\alpha_j z)] + C_{4j} \alpha_j^2 \cosh(\alpha_j z) \} \int_0^a r J_1^2(\alpha_j r) dr \]

\[+ \sum_{i=1}^{\infty} \gamma k_i^2 \cos(k_i z) \{ A_{1i} \int_0^a r I_1(k_i r) J_1(\alpha_j r) dr + A_{2i} k_i \int_0^a r^2 I_0(k_i r) J_1(\alpha_j r) dr \} \] (3.54)

\[- \gamma e^{-\alpha_j L} \{ C_{1j} [\alpha_j \cosh(\alpha_j z) + \alpha_j^2 z \sinh(\alpha_j z)] + C_{2j} \alpha_j^2 \cosh(\alpha_j z) \} \int_0^a r J_1^2(\alpha_j r) dr \]

\[- [2F_5 - \gamma 2F_1] \int_0^a r^2 J_1(\alpha_j r) dr = 0 \quad z = l; \ j = 1, 2, 3, \ldots \]

Similar to the way we found eqns (3.53) and (3.54), another two sets of equations can be generated by substituting the expressions for axial and shear stresses into eqns (3.12) and (3.13). These two sets of equations can be expressed as:
\[
\sum_{i=1}^{\infty} A_{3i} \frac{k_i^3}{r} \cos(k_i z) \int_0^a r J_0(k_i r) J_0(\alpha_j r) dr \\
+ \sum_{i=1}^{\infty} A_{4i} \frac{k_i^3}{r} \cos(k_i z) \int_0^a [(4 - 2\nu_m) \frac{\bar{F}_0}{r} + \frac{\bar{F}_1}{r}] r J_0(\alpha_j r) dr \\
+C_{3j} e^{-\alpha_j L} \alpha_j^2 [1 - 2\nu_m \cosh(\alpha_j z) - \alpha_j z \sinh(\alpha_j z)] \int_0^a r J_0^2(\alpha_j r) dr \\
-C_{4j} e^{-\alpha_j L} \alpha_j^3 \cosh(\alpha_j z) \int_0^a r J_0^2(\alpha_j r) dr \\
- \sum_{i=1}^{\infty} A_{1i} \frac{k_i^3}{r} \cos(k_i z) \int_0^a r J_0(k_i r) J_0(\alpha_j r) dr \\
+ \sum_{i=1}^{\infty} A_{2i} \frac{k_i^3}{r} \cos(k_i z) \int_0^a [(4 - 2\nu_f) J_0(k_i r) + k_i r J_1(k_i r)] r J_0(\alpha_j r) dr \\
-C_{1j} e^{-\alpha_j L} \alpha_j^2 [1 - 2\nu_f \cosh(\alpha_j z) - \alpha_j z \sinh(\alpha_j z)] \int_0^a r J_0^2(\alpha_j r) dr \\
-C_{2j} e^{-\alpha_j L} \alpha_j^3 \cosh(\alpha_j z) \int_0^a r J_0^2(\alpha_j r) dr = 0 \quad z = l; \quad j = 1, 2, 3, \ldots
\]

\[e^{-\alpha_j L} \alpha_j^2 \{C_{3j} [2\nu_m \sinh(\alpha_j z) + \alpha_j z \cosh(\alpha_j z)] + C_{4j} \alpha_j \sinh(\alpha_j z)\} \int_0^a r J_0^2(\alpha_j r) dr \\
- \sum_{i=1}^{\infty} k_i^3 \sin(k_i z) A_{1i} \int_0^a r I_1(k_i r) J_1(\alpha_j r) dr \\
+ \sum_{i=1}^{\infty} k_i^3 \sin(k_i z) A_{2i} \int_0^a [(2 - 2\nu_f) I_1(k_i r) + k_i r I_0(k_i r)] r J_1(\alpha_j r) dr \\
e^{-\alpha_j L} \alpha_j^2 \{C_{1j} [2\nu_f \sinh(\alpha_j z) + \alpha_j z \cosh(\alpha_j z)] + C_{2j} \alpha_j \sinh(\alpha_j z)\} \int_0^a r J_1^2(\alpha_j r) dr \\
= 0 \quad z = l; \quad j = 1, 2, 3, \ldots
\]

Integrating eqns (3.16), and (3.17) over the interval 0 ≤ r ≤ a with respect to r, yields

the following equations.
\[
\sum_{i=1}^{\infty} k_i^2 \sin(k_i z) \left\{ A_{1i} \int_0^a I_1(k_i r) \, dr + A_{2i} \int_0^a [(2 - 2\nu_f) I_1(k_i r) + k_i r I_0(k_i r)] \, dr \right\} + \sum_{k=1}^{\infty} C_{1k} e^{-\alpha_k L} \alpha_k^2 \left[ 2\nu_f \sinh(\alpha_k z) + \alpha_k z \cosh(\alpha_k z) \right] \int_0^a J_1(\alpha_k r) \, dr \\
+ \sum_{k=1}^{\infty} C_{2k} e^{-\alpha_k L} \alpha_k^3 \sinh(\alpha_k z) \int_0^a J_1(\alpha_k r) \, dr = 0 \quad z = L
\]

\[
\sum_{i=1}^{\infty} k_i^3 \cos(k_i z) \left\{ A_{1i} \int_0^a r I_0(k_i r) \, dr + A_{2i} \int_0^a [(4 - 2\nu_f) r I_0(k_i r) + k_i r^2 I_1(k_i r)] \, dr \right\} + \sum_{k=1}^{\infty} C_{1k} e^{-\alpha_k L} \alpha_k^2 \left[ (1 - 2\nu_f) \cosh(\alpha_k z) - \alpha_k z \sinh(\alpha_k z) \right] \int_0^a r J_0(\alpha_k r) \, dr \\
- \sum_{k=1}^{\infty} C_{2k} e^{-\alpha_k L} \alpha_k^3 \cosh(\alpha_k z) \int_0^a r J_0(\alpha_k r) \, dr \\
+ [4F_1(2 - \nu_f) + 6F_3(1 - \nu_f)] \int_0^a r \, dr = \frac{f}{2\pi} \quad z = L
\]

where \( f \) is the applied load.

Substituting the expressions for stresses into eqns (3.18) and (3.19), multiplying by \( r X_0(\lambda_j r) \) and \( r X_1(\lambda_j r) \) respectively, and integrating over the interval \( a \leq r \leq b \) with respect to \( r \), yields

\[
\sum_{i=1}^{\infty} k_i^2 \cos(k_i z) \left\{ B_{1i} \int_a^b r \tilde{K}_0 X_0 \, dr - B_{2i} \int_a^b r [(4 - 2\nu_m) \tilde{K}_0 - \tilde{k}_i r \tilde{K}_1] X_0 \, dr \right\} + \sum_{i=1}^{\infty} k_i^3 \cos(k_i z) \left\{ B_{3i} \int_a^b r \tilde{I}_0 X_0 \, dr + B_{4i} \int_a^b r [(4 - 2\nu_m) \tilde{I}_0 + \tilde{k}_i r \tilde{I}_1] X_0 \, dr \right\} \\
+ D_{1j} e^{-\lambda_j L} \lambda_j^2 \left[ (1 - 2\nu_m) \cosh(\lambda_j z) - \lambda_j z \sinh(\lambda_j z) \right] \int_a^b r X_0^2 \, dr \\
+ D_{2j} e^{-\lambda_j L} \lambda_j^3 \cosh(\lambda_j z) \int_a^b r X_0^2 \, dr = 0 \quad z = L \quad j = 1, 2, 3, \ldots.
\]
\[ +D_{ij}[2\nu_{m} \sinh(\lambda_{j}z) + \lambda_{j}z \cosh(\lambda_{j}z)] + D_{2j}\lambda_{j} \sinh(\lambda_{j}z) \]
\[ = 0 \quad z = L ; \quad j = 1, 2, 3, \ldots, j_{\text{term}} \]  

(3.60)

Two more equations are needed to provide enough equations to solve the linear system.

Substituting the expressions for displacements, eqns (3.3) and (3.4) and integrating from 0 to \( l \), yields

\[
\sum_{i=1}^{\infty} \left\{ B_{1i}k^{2}K_{0} + B_{2i}[(4 - 4\nu_{m})k^{2}K_{0} + k^{3}rK_{1}] \right\} \int_{0}^{l} \sin(k_{i}z) \, dz
\]
\[+ \sum_{i=1}^{\infty} \left\{ B_{3i}k^{2}I_{0} + B_{4i}[(4 - 4\nu_{m})k^{2}I_{0} + k^{3}rI_{1}] \right\} \int_{0}^{l} \sin(k_{i}z) \, dz
\]
\[+ \sum_{i=1}^{\infty} D_{1i}e^{-\lambda_{k}L}X_{0} \int_{0}^{l} [(2 - 4\nu_{m})\lambda_{k} \sinh(\lambda_{k}z) - \lambda_{k}^{2}z \cosh(\lambda_{k}z)] \, dz
\]
\[+ \sum_{i=1}^{\infty} D_{2i}e^{-\lambda_{k}L}k^{2}X_{0} \int_{0}^{l} \sin(\lambda_{k}z) \, dz + [8G_{1}(1 - \nu_{m}) + 6G_{3}(1 - 2\nu_{m})] \int_{0}^{l} z \, dz
\]
\[= 0 \quad r = a \]  

(3.61)

\[
\sum_{i=1}^{\infty} \left\{ A_{3i}k^{2}K_{0} + A_{4i}k^{2}I_{0} \right\} \int_{0}^{l} \sin(k_{i}z) \, dz
\]
\[+ \sum_{i=1}^{\infty} C_{3i}e^{-\alpha_{k}L}J_{0} \int_{0}^{l} [(2 - 4\nu_{m})\alpha_{k} \sinh(\alpha_{k}z) - \alpha_{k}^{2}z \cosh(\alpha_{k}z)] \, dz
\]
\[+ \sum_{i=1}^{\infty} C_{4i}e^{-\alpha_{k}L}J_{0} \alpha_{k} \int_{0}^{l} \sinh(\alpha_{k}z) \, dz - [8F_{5}(1 - \nu_{m}) + 6F_{7}(1 - 2\nu_{m})] \int_{0}^{l} z \, dz
\]
\[= 0 \quad r = a \]  

(3.62)

Equations (3.62)-(3.62) represent a complete linear system with theoretically infinite equations and infinite unknowns. The actual number of unknowns or equations is depend on how many terms were used in equations (3.33)-(3.35), the Love's stress functions. Note that all the integration terms in the equations can be integrated to give a close form solution.

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Thus the linear system is analytical. The system can then be truncated and solved by computer. It should be pointed out that the equations may looks lengthy but the computer code for the linear system can be generated systematically. The solver for a linear system is just a standard library routines and can be found in public-domain software packages. The values for the Bessel functions can also be evaluated by standard library routines from numerical packages. After the unknowns, i.e., the coefficients of the Love's stress functions, were solved. One can easily substitute the coefficient values into the expressions for the stresses and displacements to calculate stresses or displacements.

3.3 References


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3.4 Appendix

stresses in fiber (component \( f \))

\[
\sigma_f^i = \sum_{i=1}^{\infty} \left[ A_{1i} I_0(k_i r) + \frac{1}{r} I_1(k_i r) \right] \cos(k_i z) - \sum_{j=1}^{\infty} C_{1j} e^{-\alpha_j L} \left[ \alpha_j^2 ((1 - 2\nu_f) \cosh(\alpha_j z) + \alpha_j z \sinh(\alpha_j z)) J_0(\alpha_j r) - \frac{\alpha_j}{r} (\cosh(\alpha_j z) + \alpha_j z \sinh(\alpha_j z)) J_1(\alpha_j r) \right] + \sum_{j=1}^{\infty} C_{2j} e^{-\alpha_j L} \left[ -\alpha_j^3 \cosh(\alpha_j z) J_0(\alpha_j r) + \frac{\alpha_j^2}{r} \cosh(\alpha_j z) J_1(\alpha_j r) \right] - (2 - 4\nu_f) F_1 + 6\nu_f F_3
\]

\[
\sigma_f^\theta = \sum_{i=1}^{\infty} \left[ -A_{1i} \frac{k_i^2}{r} I_1(k_i r) - A_{2i} k_i^3 (1 - 2\nu_f) I_0(k_i r) \right] \cos(k_i z) + \sum_{j=1}^{\infty} C_{1j} e^{-\alpha_j L} \left[ 2\nu_f \alpha_j^2 \cosh(\alpha_j z) J_0(\alpha_j r) + \left( \frac{\alpha_j}{r} \cosh(\alpha_j z) + \frac{\alpha_j^2}{r} z \sinh(\alpha_j z) \right) J_1(\alpha_j r) \right] + \sum_{j=1}^{\infty} C_{2j} e^{-\alpha_j L} \frac{\alpha_j^2}{r} \cosh(\alpha_j z) J_1(\alpha_j r) - (2 - 4\nu_f) F_1 + 6\nu_f F_3
\]

\[
\sigma_f^z = \sum_{i=1}^{\infty} k_i^3 \left[ A_{1i} I_1(k_i r) + A_{2i} \left( (4 - 2\nu_f) I_0(k_i r) + k_i r I_1(k_i r) \right) \right] \cos(k_i z) + \sum_{j=1}^{\infty} C_{1j} e^{-\alpha_j L} \alpha_j^2 \left[ (1 - 2\nu_f) \cosh(\alpha_j z) - \alpha_j z \sinh(\alpha_j z) \right] J_0(\alpha_j r) - \sum_{j=1}^{\infty} C_{2j} e^{-\alpha_j L} \alpha_j^3 \cosh(\alpha_j z) J_0(\alpha_j r) + 4(2 - \nu_f) F_1 + 6(1 - \nu_f) F_3
\]

\[
\tau_{r_z}^f = \sum_{i=1}^{\infty} k_i^3 \left[ A_{1i} I_1(k_i r) + A_{2i} \left( (2 - 2\nu_f) I_1(k_i r) + k_i r I_0(k_i r) \right) \right] \sin(k_i z) + \sum_{j=1}^{\infty} C_{1j} e^{-\alpha_j L} \alpha_j^2 \left[ 2\nu_f \sinh(\alpha_j z) + \alpha_j z \cosh(\alpha_j z) \right] J_1(\alpha_j r) + \sum_{j=1}^{\infty} C_{2j} e^{-\alpha_j L} \alpha_j^3 \sinh(\alpha_j z) J_1(\alpha_j r)
\]
\[
2G'U'_t = \sum_{i=1}^{\infty} k_i^2 \left[ -A_{1i}I_1(\k_i r) - A_{2i}k_i rI_0(\k_i r) \right] \cos(\k_i z) \\
+ \sum_{j=1}^{\infty} C_{1j}e^{-\alpha_j L}\k_j \left[ \cosh(\alpha_j z) + \alpha_j z \sinh(\alpha_j z) \right] J_1(\alpha_j r) \\
+ \sum_{j=1}^{\infty} C_{2j}e^{-\alpha_j L}\alpha_j^2 \cosh(\alpha_j z)J_1(\alpha_j r) - 2F_1 r
\]

\[
2G'U'_z = \sum_{i=1}^{\infty} k_i^2 \left[ A_{1i}I_0(\k_i r) + A_{2i} \left( (4 - 4\nu_f)I_0(\k_i r) + k_i rI_1(\k_i r) \right) \right] \sin(\k_i z) \\
+ \sum_{j=1}^{\infty} C_{1j}e^{-\alpha_j L}\k_j \left[ (2 - 4\nu_f)\sinh(\alpha_j z) - \alpha_j z \cosh(\alpha_j z) \right] J_0(\alpha_j r) \\
- \sum_{j=1}^{\infty} C_{2j}e^{-\alpha_j L}\alpha_j^2 \sinh(\alpha_j z)J_0(\alpha_j r) + 8(1 - \nu_f)F_1 z + 6(1 - 2\nu_f)F_3 z
\]

\textit{stresses in matrix (component } m_1 \text{)}

\[
\sigma_{r}^{m_1} = \sum_{i=1}^{\infty} \left[ -B_{1i}k_i^2 \left( \k_i K_0(\k_i r) + \frac{1}{r} K_1(\k_i r) \right) + B_{2i}k_i^3 \left( (1 - 2\nu_m)K_0(\k_i r) - \k_i r K_1(\k_i r) \right) \right] \cos(\k_i z) \\
+ \sum_{i=1}^{\infty} \left[ -B_{3i}k_i^2 \left( \k_i I_0(\k_i r) - \frac{1}{r} I_1(\k_i r) \right) + B_{4i}k_i^3 \left( (1 + 2\nu_m)I_0(\k_i r) - \k_i r I_1(\k_i r) \right) \right] \cos(\k_i z) \\
+ \sum_{j=1}^{\infty} D_{1j}e^{-\lambda_j L} \left\{ \cosh(\lambda_j z) + \lambda_j z \sinh(\lambda_j z) \right\} \left[ \lambda_j^2 \left( J_0(\lambda_j r) + \mu_j Y_0(\lambda_j r) \right) - \frac{\lambda_j}{r} \left( J_1(\lambda_j r) + \mu_j Y_1(\lambda_j r) \right) \right] \\
+ 2\nu_m \lambda_j^2 \cosh(\lambda_j z) \left( J_0(\lambda_j r) + \mu_j Y_0(\lambda_j r) \right)) \\
+ \sum_{j=1}^{\infty} D_{2j}e^{-\lambda_j L}\lambda_j \cosh(\lambda_j z) \left[ \lambda_j^2 \left( J_0(\lambda_j r) + \mu_j Y_0(\lambda_j r) \right) - \frac{\lambda_j}{r} \left( J_1(\lambda_j r) + \mu_j Y_1(\lambda_j r) \right) \right] \\
- G_1(2 - 4\nu_m) + \frac{G_2}{r^2} + 6\nu_m G_3
\]
\[ \sigma_{\mu 1}^m = \sum_{i=1}^{\infty} \left[ \frac{k_i^2}{r} K_1(k_ir) + B_2i k_i^3 (1 - 2\nu_m)K_0(k_ir) - B_3i k_i^3 I_1(k_ir) - B_4i k_i^3 (1 - 2\nu_m)I_0(k_ir) \right] \cos(k_iz) \\
+ \sum_{j=1}^{\infty} D_{1j} e^{-\lambda_j L} \{ 2\nu_m \lambda_j^2 \cosh(\lambda_j z) \left[ J_0(\lambda_j r) + \mu_j Y_0(\lambda_j r) \right] + \frac{\lambda_j}{r} \cosh(\lambda_j z) \left[ J_1(\lambda_j r) + \mu_j Y_1(\lambda_j r) \right] \\
+ \frac{\lambda_j^2}{r} z \sinh(\lambda_j z) \left[ J_1(\lambda_j r) + \mu_j Y_1(\lambda_j r) \right] \} \\
+ \sum_{j=1}^{\infty} D_{2j} e^{-\lambda_j L} \frac{\lambda_j^2}{r} \cosh(\lambda_j z) \left[ J_0(\lambda_j r) + \mu_j Y_0(\lambda_j r) \right] - G_1(2 - 4\nu_m) - \frac{G_2}{r^2} + 6\nu_m G_3 \]

\[ \sigma_{z 1}^m = \sum_{i=1}^{\infty} \left[ \frac{k_i^2}{r} K_0(k_i r) - B_2i k_i^3 (1 - 2\nu_m)K_0(k_i r) - k_i r K_1(k_i r) \right] \cos(k_iz) \\
+ \sum_{i=1}^{\infty} \left[ B_{3i} k_i^3 I_0(k_i r) + B_{4i} k_i^3 ((4 - 2\nu_m)I_0(k_i r) + k_i r I_1(k_i r)) \right] \cos(k_iz) \\
+ \sum_{j=1}^{\infty} D_{1j} e^{-\lambda_j L} \{ (1 - 2\nu_m)\lambda_j^3 \cosh(\lambda_j z) - \lambda_j^3 z \sinh(\lambda_j z) \left[ J_0(\lambda_j r) + \mu_j Y_0(\lambda_j r) \right] \\
+ \sum_{j=1}^{\infty} D_{2j} e^{-\lambda_j L} \lambda_j^3 \cosh(\lambda_j z) \left[ J_0(\lambda_j r) + \mu_j Y_0(\lambda_j r) \right] + 4G_1(2 - \nu_m) + 6(1 - \nu_m)G_3 \]
\[2G^m U_{r}^{m1} = \sum_{i=1}^{\infty} \left( B_{4i} \bar{k}_i r + B_{A2} \bar{k}_i r K_0(\bar{k}_i r) - B_{3i} I_1(\bar{k}_i r) - B_{4i} \bar{k}_i r I_0(\bar{k}_i r) \right) \cos(\bar{k}_i z) \]
\[+ \sum_{j=1}^{\infty} D_{1j} e^{-\lambda_j L} \left\{ \lambda_j^2 z \sinh(\lambda_j z) + \lambda_j \cosh(\lambda_j z) \right\} \left[ J_1(\lambda_j r) + \mu_j Y_1(\lambda_j r) \right] \]
\[+ \sum_{j=1}^{\infty} D_{2j} e^{-\lambda_j L} \lambda_j^2 \cosh(\lambda_j z) \left[ J_0(\lambda_j r) + \mu_j Y_0(\lambda_j r) \right] - 2G_1 r - \frac{G_2}{r} \]
\[2G^m U_{x}^{m1} = \sum_{i=1}^{\infty} \left[ B_{1i} \bar{k}_i^2 K_0(\bar{k}_i r) + B_{2i} \bar{k}_i^2 \left( (-4 + 4\nu_m)K_0(\bar{k}_i r) + \bar{k}_i r K_1(\bar{k}_i r) \right) \right] \sin(\bar{k}_i z) \]
\[+ \sum_{i=1}^{\infty} \left[ B_{3i} \bar{k}_i^2 I_0(\bar{k}_i r) + B_{4i} \bar{k}_i^2 \left( (4 - 4\nu_m)I_0(\bar{k}_i r) + \bar{k}_i r I_1(\bar{k}_i r) \right) \right] \sin(\bar{k}_i z) \]
\[+ \sum_{j=1}^{\infty} D_{1j} e^{-\lambda_j L} \{ (2 - 4\nu_m)\lambda_j \sinh(\lambda_j z) - \lambda_j^2 z \cosh(\lambda_j z) \} \left[ J_0(\lambda_j r) + \mu_j Y_0(\lambda_j r) \right] \]
\[+ \sum_{j=1}^{\infty} D_{2j} e^{-\lambda_j L} \lambda_j^2 \sin(\lambda_j z) \left[ J_0(\lambda_j r) + \mu_j Y_0(\lambda_j r) \right] + 8G_1 z(1 - \nu_m) + 6G_3 z(1 - 2\nu_m) \]

stresses in matrix (component \( m_2 \))
\[\sigma_{r}^{m2} = \sum_{i=1}^{\infty} \left[ A_{3i} \bar{k}_i \left( -\bar{k}_i I_0(\bar{k}_i r) + \frac{1}{r} I_1(\bar{k}_i r) \right) - A_{4i} \bar{k}_i \left( (1 - 2\nu_m)I_0(\bar{k}_i r) + \bar{k}_i r I_1(\bar{k}_i r) \right) \right] \cos(\bar{k}_i z) \]
\[+ \sum_{j=1}^{\infty} C_{3j} e^{-\alpha_j L} \left[ \alpha_j^2 ((1 - 2\nu_m) \cosh(\alpha_j z) + \alpha_j \sinh(\alpha_j z)) J_0(\alpha_j r) - \frac{\alpha_j}{r} (\cosh(\alpha_j z) + \alpha_j \sinh(\alpha_j z)) J_1(\alpha_j r) \right] \]
\[+ \sum_{j=1}^{\infty} C_{4j} e^{-\alpha_j L} \left[ -\alpha_j^3 \sinh(\alpha_j z) J_0(\alpha_j r) + \frac{\alpha_j^2}{r} \cosh(\alpha_j z) J_1(\alpha_j r) \right] \]
\[- (2 - 4\nu_m) F_5 + 6\nu_m F_7 \]
\[\sigma_{\theta}^{m2} = \sum_{i=1}^{\infty} \left[ -A_{3i} \bar{k}_i \frac{\bar{k}_i}{r} I_1(\bar{k}_i r) - A_{4i} \bar{k}_i \left( (1 - 2\nu_m)I_0(\bar{k}_i r) \right) \right] \cos(\bar{k}_i z) \]
\[+ \sum_{j=1}^{\infty} C_{3j} e^{-\alpha_j L} \left[ 2\nu_m \alpha_j^3 \cosh(\alpha_j z) J_0(\alpha_j r) + \left( \frac{\alpha_j}{r} \cosh(\alpha_j z) + \frac{\alpha_j^2}{r} \sinh(\alpha_j z) \right) J_1(\alpha_j r) \right] \]
\[+ \sum_{j=1}^{\infty} C_{4j} e^{-\alpha_j L} \frac{\alpha_j^2}{r} \cosh(\alpha_j z) J_1(\alpha_j r) - (2 - 4\nu_m) F_5 + 6\nu_m F_7 \]

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\[
\alpha_{2m} = \sum_{i=1}^{\infty} \tilde{k}_{i} \left[ A_{3i}I_{0}(\tilde{k}_{i}r) + A_{4i} \left( (4 - 2\nu_{m})I_{0}(\tilde{k}_{i}r) + \tilde{k}_{i}rI_{1}(\tilde{k}_{i}r) \right) \right] \cos(\tilde{k}_{i}z) \\
+ \sum_{j=1}^{\infty} C_{3j}e^{-\alpha_{j}L} \alpha_{j}^{2} \left[ (1 - 2\nu_{m})\cosh(\alpha_{j}z) - \alpha_{j}z\sinh(\alpha_{j}z) \right] J_{0}(\alpha_{j}r) \\
- \sum_{j=1}^{\infty} C_{4j}e^{-\alpha_{j}L} \alpha_{j}^{2} \cosh(\alpha_{j}z)J_{0}(\alpha_{j}r) + 4(2 - \nu_{m})F_{5} + 6(1 - \nu_{m})F_{7} \\
\beta_{2m} = \sum_{i=1}^{\infty} \tilde{k}_{i} \left[ A_{3i}I_{1}(\tilde{k}_{i}r) + A_{4i} \left( (2 - 2\nu_{m})I_{1}(\tilde{k}_{i}r) + \tilde{k}_{i}rI_{0}(\tilde{k}_{i}r) \right) \right] \sin(\tilde{k}_{i}z) \\
+ \sum_{j=1}^{\infty} C_{3j}e^{-\alpha_{j}L} \alpha_{j}^{2} \left[ 2\nu_{m}\sinh(\alpha_{j}z) + \alpha_{j}z\cosh(\alpha_{j}z) \right] J_{1}(\alpha_{j}r) \\
+ \sum_{j=1}^{\infty} C_{4j}e^{-\alpha_{j}L} \alpha_{j}^{2} \sinh(\alpha_{j}z)J_{1}(\alpha_{j}r) \\
2G^{m}U_{r}^{m2} = \sum_{i=1}^{\infty} \tilde{k}_{i} \left[ -A_{3i}I_{1}(\tilde{k}_{i}r) - A_{4i} \tilde{k}_{i}rI_{0}(\tilde{k}_{i}r) \right] \cos(\tilde{k}_{i}z) \\
+ \sum_{j=1}^{\infty} C_{3j}e^{-\alpha_{j}L} \alpha_{j} \left[ \cosh(\alpha_{j}z) + \alpha_{j}z\sinh(\alpha_{j}z) \right] J_{1}(\alpha_{j}r) \\
+ \sum_{j=1}^{\infty} C_{4j}e^{-\alpha_{j}L} \alpha_{j}^{2} \cosh(\alpha_{j}z)J_{1}(\alpha_{j}r) - 2F_{5}r \\
2G^{m}U_{z}^{m2} = \sum_{i=1}^{\infty} \tilde{k}_{i} \left[ A_{3i}I_{0}(\tilde{k}_{i}r) + A_{4i} \left( (4 - 4\nu_{m})I_{0}(\tilde{k}_{i}r) + \tilde{k}_{i}rI_{1}(\tilde{k}_{i}r) \right) \right] \sin(\tilde{k}_{i}z) \\
+ \sum_{j=1}^{\infty} C_{3j}e^{-\alpha_{j}L} \alpha_{j} \left[ (2 - 4\nu_{m})\sinh(\alpha_{j}z) - \alpha_{j}z\cosh(\alpha_{j}z) \right] J_{0}(\alpha_{j}r) \\
- \sum_{j=1}^{\infty} C_{4j}e^{-\alpha_{j}L} \alpha_{j}^{2} \sinh(\alpha_{j}z)J_{0}(\alpha_{j}r) + 8(1 - \nu_{m})F_{5}z + 6(1 - 2\nu_{m})F_{7}z
\]
Fig. 3.1 The 3-D single fiber pull out model
Fig. 3.2 Scheme of the axisymmetric boundary value problem
Micromechanics and Effective Transverse Elastic Moduli of Composites with Randomly Located Aligned Circular Fibers

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Abstract

Based on the two-dimensional (plane strain) micromechanical fiber interaction framework, effective transverse elastic moduli of two-phase brittle matrix composites containing many randomly located yet unidirectionally aligned circular fibers are investigated in this paper. The fibers are characterized as infinitely long and equal-sized inclusions. By employing the local pairwise fiber interaction formulation coupled with the ensemble-area averaged field equations, the proposed approximate analysis leads to a novel, higher-order (in $\phi$), and accurate method for the prediction of effective transverse elastic moduli of two-phase fiber reinforced composites. In addition, the proposed micromechanical approach is extended to predict the effective transverse shear viscosities of fiber suspensions with randomly located aligned rigid fibers. Comparisons with experimental data, Hashin's variational bounds, and improved three-point bounds are also presented to illustrate the predictive capability of the proposed method for fiber-reinforced composites.

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1. Introduction

Fiber reinforced composites have long been of research interests due to their better and tailored mechanical performance over traditional materials as well as other advantages such as weight savings, high stiffness/weight and strength/weight ratios, better environmental durability and resistance against corrosion and humidity, and so on. Applications of fiber reinforced composites have been made to aircrafts, space shuttles, automobiles, sporting goods, and civil engineering structures (cf. Mallick, 1993). Fiber reinforced composites can be described as a matrix material reinforced by fibers of another material. The fibers could be short or long, aligned or randomly oriented, and periodic or randomly dispersed. The prediction and estimation of effective mechanical and conductive properties of fiber reinforced composites are therefore of great interests to engineers and researchers.

In this paper, we consider a linearly elastic isotropic matrix reinforced by linearly elastic, unidirectionally aligned, randomly located, impenetrable, and infinitely long circular fibers. The fibers could be isotropic or transversely isotropic. Furthermore, we assume that the composite specimen is: (a) statistically homogeneous and statistically transversely isotropic (Hashin, 1965; Ju and Chen, 1994a, 1994b); and (b) with perfect interfacial bonding between the matrix and fibers. The overall composite is therefore transversely isotropic. As shown in Figure 1, the Cartesian coordinate system can be set up with axis-3 parallel to the fiber direction. The overall transverse isotropy can be characterized by five effective elastic moduli; namely, the plane-strain bulk modulus $\kappa^*$, the transverse shear modulus $\mu_T^*$, the axial Young's modulus $E_A^*$, the axial Poisson's ratio $\nu_A^*$ and the axial shear modulus $\mu_A^*$. Hill (1964) showed that there were only three independent effective elastic constants for such fibrous composites and the other two constants could be easily determined.

Many theoretical methods have been developed in the literature to predict effective elastic moduli of fiber reinforced composites; see Hashin (1983), Mura (1987), Zhao and Weng (1990), and Nemat-Nasser and Hori (1993) for some literature reviews. The first school, stemming from the pioneering work of Hashin and Rosen (1964) and Hill (1964), employed variational principles or linear comparison composites to obtain mathematical lower and upper bounds for effective elastic, transversely isotropic moduli of fiber reinforced composites. Subsequently, Hashin (1965) derived the variational upper and lower bounds for the plane strain bulk modulus $\kappa^*$, and the transverse and axial shear moduli $\mu_T^*$ and $\mu_A^*$ of fiber reinforced composites with arbitrary transverse phase geometry in terms of constituent phase elastic moduli and phase volume fractions. Upper
and lower bounds on effective axial Young's modulus $E_A^*$ and axial Poisson's ratio $\nu_A^*$ were later derived by Hashin (1972). “Improved” third-order mathematical bounds, which depend on statistical microstructural informations of random fiber composites, were derived by Silnutzer (1972) for $\mu_A^*$, $\kappa^*$ and $\mu_T^*$. Moreover, Milton (1982) proposed the fourth-order bounds for the effective axial shear modulus $\mu_A^*$; see also Torquato and Lado (1992) for detailed calculations on the third-order and fourth-order bounds. Nomura and Chou (1984) also proposed the third-order bounds based on variational principles and the three-point correlation functions. It is noted that the third-order bounds are narrower than the two-point bounds of Hashin's type.

The second school for micromechanical estimation of effective moduli of fiber reinforced composites is known as the effective medium approach. This category includes the self-consistent method (Kröner, 1958; Budiansky, 1965; Hill, 1965), the differential scheme (McLaughlin, 1977; Hashin, 1988), the generalized self-consistent method (Christensen and Lo, 1979; Christensen, 1990), and the Mori-Tanaka method (Mori and Tanaka, 1973; Taya and Mura, 1981; Taya, 1981; Weng, 1984, 1990; Qiu and Weng, 1990). Nevertheless, effective medium methods do not depend on particle locations or their relative configurations. By contrast, the third school aims at direct micromechanical determination of effective properties of composites with randomly located and interacting inclusions by employing some approximations, or with certain special geometric configurations for inclusions dispersing in matrix materials. For example, the second-order formulations with pairwise inter-particle interactions were proposed in the 1970s for perfectly randomly dispersed spherical particles. Further, a micromechanical higher-order ensemble-volume average formulation was proposed by Ju and Chen (1994a, 1994b) for isotropically randomly located spherical particles. For aligned fiber reinforced composites, however, no such work was proposed along the line of the third school. Only local (not overall) plane-strain fiber interactions were investigated in the literature; see, e.g., Shioya (1971), Kouris and Tsuchida (1991), Honein (1991), and Honein et al. (1992, 1994a, 1994b).

In this paper, we attempt to construct an approximate yet accurate method to account for inter-fiber interaction effects in fiber reinforced composites. In combination with the ensemble-area averaged field equations, this work presents a new, higher-order (in fiber volume fraction $\phi$), probabilistic approach to estimate effective transverse elastic moduli of two-phase composites containing unidirectionally aligned yet randomly located circular fibers.

This paper is organized as follows. In Section 2, approximate solutions for the local interaction problem of two identical yet randomly located elastic circular fibers embedded in an elastic
Effective Transverse Elastic Moduli of Fiber Reinforced Composites

matrix are presented. Subsequently, the ensemble-area averaged eigenstrain is obtained through probabilistic pairwise fiber interaction mechanism in Section 3. Both uniform and general radial distribution functions are considered. Combining the results from Sections 2–3 and the governing ensemble-area averaged field equations, effective transversely isotropic elastic moduli of fiber reinforced composites are derived in Section 4. It is found that our “noninteracting solutions” (obtained by dropping the pairwise fiber interactions) are identical to the variational bounds of Hashin (1965). In addition, comparisons and simulations between our predictions and other methods as well as experimental data are given in Section 5. By the analogy between the effective transverse shear modulus and the effective transverse shear viscosity, we extend our predictions to fiber suspensions with rigid fibers and an incompressible viscous fluid matrix.

2. Approximate local solutions of two interacting circular fibers

Shioya (1971) proposed an analysis for an infinitely large, thin plate containing a pair of circular elastic inhomogeneities and subjected to uniform tensions. Shioya's method was based on the Airy’s stress function in the generalized plane stress and employed the bipolar coordinates together with a method of perturbation. Kouris and Tsuchida (1991) presented an analytical method to solve the elastic interaction problem between two circular fibers under the plane strain thermal loading. Their method was based on the displacement potential approach. Furthermore, Honéin (1991, Chap. 7) and Honein et al. (1994a, 1994b) proposed a general framework to solve the problem of two circular inclusions in plane elastostatics, subjected to arbitrary loading. Honein's approach hinged on the use of the Kolosov-Muskhelishvili complex potentials. Although the aforementioned methods are valuable, they are computationally involved and require numerical calculations of many terms in some infinite series.

In this section, instead of trying to derive exact local solutions of the randomly located two circular fibers interaction problem, we attempt to construct simple and accurate approximate analytical solutions. The proposed local approximate analytical solutions are quite compact and amenable to the pairwise ensemble-area average approach (to be explained in Section 3), leading to accurate estimates of effective elastic transverse moduli of two-phase composites containing many randomly located yet unidirectionally aligned circular fibers at moderately high volume fractions. For mathematical simplicity, we will assume that circular fibers are of equal size in what follows.
Let us consider two identical, unidirectionally aligned elastic circular fibers (phase 1) of radius \(a\) embedded in a homogeneous elastic matrix (phase 0). Since plane strain is assumed, the inclusion interaction exists only in the same cutting plane as shown in Figure 2. In addition, the plane-strain linearly elastic isotropic stiffness tensors for both phases are denoted by

\[
(C_{\alpha})_{ijkl} = \lambda_\alpha \delta_{ij} \delta_{kl} + \mu_\alpha (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad \alpha = 0, 1; \quad i, j, k, l = 1, 2
\]

(1)

where \(\lambda_\alpha\) and \(\mu_\alpha\) are the Lamé constants of the phase \(\alpha\) material.

Following the eigenstrain concept introduced by Eshelby (1957), an integral equation governing the distributed eigenstrain \(\epsilon^*(x)\) for a given particle (fiber) configuration and remote strain field \(\epsilon^o\) was derived in Eq. (7) in Ju and Chen (1994a). Within the present two-fiber context, the integral equation can be rephrased as follows:

\[
-A : \epsilon^*(x) = \epsilon^o + \int_{\Omega_1} G(x - x') : \epsilon^*_{(1)}(x') \, dx' \\
+ \int_{\Omega_2} G(x - x') : \epsilon^*_{(2)}(x') \, dx', \quad i \neq j, \quad i, j = 1, 2
\]

(2)

where \(x \in \Omega_i\) and \(\epsilon^*_{(i)}(x)\) is the eigenstrain at \(x\) in the \(i\)-th fiber domain \(\Omega_i\). In addition, the fourth-rank tensor \(A\) is defined as

\[
A \equiv (C_1 - C_0)^{-1} \cdot C_0
\]

(3)

and the components of the fourth-rank two-dimensional Green's function tensor \(G\) are given by \((i, j, k, l = 1, 2; \text{cf. Mura, 1987})\):

\[
G_{ijkl}(x - x') = \frac{1}{4\pi(1 - \nu_0) \|r'\|^2} F_{ijkl}(-8, 2\nu_0, 2, 2 - 4\nu_0, -1 + 2\nu_0, 1 - 2\nu_0)
\]

(4)

where \(r' \equiv x - x'\) and \(r' \equiv \|r'\|\). The components of the fourth-rank tensor \(F\) — which depends on its arguments \((B_1, B_2, B_3, B_4, B_5, B_6)\) — are defined by \((m = 1 \text{ to } 6)\):

\[
F_{ijkl}(B_m) \equiv B_1 n_i^j n_j^k n_k^l + B_2 (\delta_{ik} n_j^l n_k^j + \delta_{il} n_j^j n_k^l + \delta_{jk} n_i^j n_k^i + \delta_{jl} n_i^i n_k^j) \\
+ B_3 \delta_{ij} n_k^i n_l^k + B_4 \delta_{kl} n_i^j n_j^i + B_5 \delta_{ij} \delta_{kl} + B_6 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
\]

(5)

with the normal vector \(n' \equiv r'/\|r'\|\). All physical quantities refer to the Cartesian coordinates, and the summation convention applies. Moreover, \(\delta_{ij}\) denotes the Kronecker delta and \(\nu_0\) is the Poisson's ratio of the matrix material.

As indicated in Ju and Chen (1994a, 1994b), the "noninteracting" solution for the eigenstrain, denoted by \(\epsilon^{*o}\), can be obtained by dropping the last term in the right-hand side of (2), which represents the interaction effect due to the other fiber. The "noninteracting" result is

\[
-A : \epsilon^{*o} = \epsilon^o + s : \epsilon^{*o}
\]

(6)
where the two-dimensional Eshelby tensor $s$ is defined as

$$s \equiv \int_{\Omega_i} G(x - x') \, dx' ; \quad x, x' \in \Omega_i$$  \hspace{1cm} (7)

The components of $s$ depend on the Poisson's ratio of the matrix ($\nu_0$) and the shape of the fiber cross-sectional domain $\Omega_i$. In particular, for a two-dimensional circular domain, the components of $s$ are

$$s_{ijkl} = \frac{1}{8(1 - \nu_0)} \left\{ (4\nu_0 - 1)\delta_{ij}\delta_{kl} + (3 - 4\nu_0)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \right\}$$  \hspace{1cm} (8)

See Mura (1987) and Appendix I for more details.

By subtracting the noninteracting solution (6) from (2), the effect of inter-fiber interaction can be found by solving the following integral equation:

$$-A : d^{*\prime}(x) = \int_{\Omega_j} G(x - x') \, dx' : \epsilon^\circ + \int_{\Omega_i} G(x - x') : d^{*\prime}(x') \, dx'$$

$$+ \int_{\Omega_j} G(x - x') : d^{*\prime\prime}(x') \, dx', \quad \text{for } x \in \Omega_i, \ i \neq j$$  \hspace{1cm} (9)

where

$$d^{*\prime}(x) \equiv \epsilon^{\prime\prime}(x) - \epsilon^\circ$$  \hspace{1cm} (10)

To obtain the higher-order interaction correction for $\epsilon^{*\prime}(x)$, one may expand the fourth-rank tensor $G(x - x')$ in the domain $\Omega_j$ with respect to its central point $x_j$; i.e.,

$$G(x - x') = G(x - x_j) - (x' - x_j) \cdot [\nabla_x \otimes G(x - x_j)]$$

$$+ \frac{1}{2}[(x' - x_j) \otimes (x' - x_j)] : [\nabla_x \otimes \nabla_x \otimes G(x - x_j)] + ...$$  \hspace{1cm} (11)

in which the relation

$$\nabla_{x'} \otimes G(x - x') = -\nabla_x \otimes G(x - x')$$  \hspace{1cm} (12)

has been used. From Eqns. (9) and (11), we have

$$-A : d^{*\prime}(x) = \int_{\Omega_j} G(x - x') \, dx' : \epsilon^\circ + \int_{\Omega_i} G(x - x') : d^{*\prime}(x') \, dx'$$

$$+ \Omega G(x - x_j) : \bar{d}^{*\prime\prime}(x_j) - \Omega a \{\nabla_x \otimes G(x - x_j) \} : \bar{P}^{*\prime\prime}$$

$$+ \frac{1}{2} \Omega a^2 \{\nabla_x \otimes \nabla_x \otimes G(x - x_j) \} : \bar{Q}^{*\prime\prime} + ...$$  \hspace{1cm} (13)

for $x \in \Omega_i$ and $i \neq j$ ($i, j = 1, 2$). Here, $\Omega = \pi a^2$ is the cross-sectional area of a single fiber, and $a$ is its radius. Furthermore, the average fields involved in (13) are defined as follows:

$$\bar{d}^{*\prime\prime}(x) \equiv \frac{1}{\Omega} \int_{\Omega_j} d^{*\prime\prime}(x) \, dx$$  \hspace{1cm} (14)
The third rank tensor $\tilde{P}^{*\langle i\rangle}$ and the fourth rank tensor $\tilde{Q}^{*\langle i\rangle}$ correspond to the dipole and quadrapole of $d^{*\langle i\rangle}$ in the domain $\Omega_j$, respectively. Due to the circular symmetry of fibers, the leading order of $\tilde{P}^{*\langle i\rangle}$ can be shown to be of the order $O(\rho^3)$, rather than $O(\rho^2)$, by substituting (13) into (15). Here, $\rho \equiv a/r$ and $r$ is the spacing between the centers of two interacting fibers. By performing the area average of (13) for the domain $\Omega_i$ and dropping those terms of higher order moments in (13), the approximate equations for $d^{*\langle i\rangle}$ for the two-fiber interaction problem can be obtained:

$$-A : \tilde{d}^{*\langle i\rangle} = G^2(x_i - x_j) : \epsilon^{*\circ} + S : \tilde{d}^{*\langle i\rangle} + G^1(x_i - x_j) : \tilde{d}^{*\langle j\rangle} + O(\rho^6)$$  

where

$$G^1 \equiv \int_{\Omega_1} G(x - x_2) \, dx = \int_{\Omega_2} G(x_1 - x) \, dx = \frac{1}{8(1 - \nu_0)} \left( \rho^2 H^1 + \frac{\rho^4}{2} H^2 \right)$$  

$$G^2 \equiv \frac{1}{\Omega} \int_{\Omega_1} \int_{\Omega_2} G(x - x') \, dx' \, dx = \frac{1}{8(1 - \nu_0)} \left( \rho^2 H^1 + \rho^4 H^2 \right)$$

and the components of $H^1$ and $H^2$ are given by

$$H^1_{ijkl}(x_1 - x_2) \equiv 2 F_{ijkl}(-8, 2\nu_0, 2, 2 - 4\nu_0, -1 + 2\nu_0, 1 - 2\nu_0)$$

$$H^2_{ijkl}(x_1 - x_2) \equiv 2 F_{ijkl}(24, -4, -4, -4, 1, 1)$$

It is interesting to note that $G^1$ in Eqn. (18) is different from the Eshelby tensor $s$ defined in (7). One may refer to $G^1$ as the "exterior-point Eshelby tensor" since the integrals in (18) involves one exterior point (e.g., $x_2$, the center of $\Omega_2$) outside the integration domain (e.g., $\Omega_1$).

It should be noted that the leading-order error induced by truncating the higher order moments in (17) is of the order $O(\rho^6)$, since both $\tilde{P}^{*\langle i\rangle}$ and $\Omega a \nabla_x \otimes G$ are of the order $O(\rho^3)$. Furthermore, we observe from (17) that

$$\tilde{d}^{*\langle i\rangle} = \tilde{d}^{*\langle 2\rangle} \equiv \tilde{d}^*$$

Therefore, the solutions of (17) are

$$\tilde{d}^* = -8(1 - \nu_0) \left[ T^{-1} \cdot G^2 \right] : \epsilon^{*\circ} + O(\rho^6)$$

where

$$T(x_1 - x_2) \equiv -8(1 - \nu_0) \{ -A - S - G^1(x_1 - x_2) \}$$
The procedure for finding the inverse of the fourth-rank tensor $T$ is given in Appendix II. The corresponding expression to the order of $O(p^2)$ is

$$T^{-1} = K^{-1} + p^2 L + O(p^4)$$

(25)

where

$$K_{ijkl} \equiv F_{ijkl}(0, 0, 0, 0, \alpha, \beta)$$

(26)

$$L_{ijkl} \equiv \frac{1}{\beta^2} F_{ijkl} \left(4, \nu_0, -2(1 - \nu_0), -2(1 - \nu_0) + \frac{\beta(1 - 2\nu_0)}{\alpha + \beta}, -2(1 - \nu_0) + \frac{\beta}{\alpha + \beta}, \frac{3 - 4\nu_0}{2}, -2(1 - \nu_0) + \frac{\beta}{\alpha + \beta}, \frac{1 - 2\nu_0}{2}\right)$$

(27)

and

$$\alpha = (4\nu_0 - 1) + 4(1 - \nu_0) \cdot \left(\frac{\kappa_0}{\kappa_1 - \kappa_0} - \frac{\mu_0}{\mu_1 - \mu_0}\right)$$

(28)

$$\beta = (3 - 4\nu_0) + 4(1 - \nu_0) \frac{\mu_0}{\mu_1 - \mu_0}$$

(29)

Here, $\kappa_0, \kappa_1$ and $\mu_0, \mu_1$ are the plane strain bulk and shear moduli of the matrix and fiber phases, respectively. Substitution of (25) into (23) then renders the final expression for $d^*$:

$$d^* = -\left[K^{-1} \cdot (p^2 H^1 + p^4 H^2)\right] \cdot \varepsilon^* - \rho^4[L \cdot H^1] \cdot \varepsilon^* + O(p^6)$$

(30)

Since $r > 2a$, we have $p < \frac{1}{2}$.

3. Ensemble-area averaged eigenstrains

In order to obtain the ensemble-average solution of $d^*$ within the context of approximate pair-wise fiber interaction, one has to integrate (30) over all possible positions ($x_2$) of the second fiber for a given location of the first fiber ($x_1$). The ensemble-average process can be written as

$$\langle d^* \rangle(x_1) = \int_{A-O_1} d^*(x_1 - x_2) P(x_2|x_1) dx_2$$

(31)

where $P(x_2|x_1)$ is the conditional probability function for finding the second fiber centered at $x_2$ given the first fiber centered at $x_1$. Moreover, angled brackets signify the ensemble (probabilistic) average operators. In this paper, we shall consider only two-dimensional, statistically transversely isotropic and homogeneous two-point probability function $P(x_2|x_1)$. The (infinitely large) 2-D transversely isotropic probabilistic (not physical) integration domain $A$ in (31) can therefore be evaluated as circular. It is noted that $\Omega_1$ in (31) defines the probabilistic "exclusion zone" for $x_2$. 
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The two-point conditional probability function \( P(x_2|x_1) \) depends on the 2-D microstructure of a composite which in turn depends on the fiber volume fraction and the underlying manufacturing processes. In the absence of actual manufacturing and microstructural evidences, it is often assumed that the 2-D two-point conditional probability function takes the following form:

\[
P(x_2|x_1) = \begin{cases} 
\frac{N}{A} g(\hat{r}), & \text{if } \hat{r} \geq 1 \\
0, & \text{otherwise}
\end{cases} \tag{32}
\]

where \( N/A \) is the 2-D number density of fibers in a composite and \( r \) is the spacing between the centers of two fibers (\( \hat{r} \equiv r/2a \)). Further, \( g(\hat{r}) \) denotes the 2-D transversely isotropic "radial distribution function".

Case I: Uniform Radial Distribution Function

In this case, we have \( g(\hat{r}) = 1 \). This corresponds to the simplest approximation for \( g(\hat{r}) \) since it tends to underestimate the probability of the second fiber \( x_2 \) in the near neighborhood of the first fiber \( x_1 \) at high fiber volume fraction. Therefore, this case may be regarded as the "lower bound" for microstructure and is more suitable for low fiber concentrations. By substituting Eqn. (30) and (32) into Eqn. (31), the explicit expression for \( \langle \tilde{\varepsilon}^* \rangle(x_1) \) can be rephrased as

\[
\langle \tilde{\varepsilon}^* \rangle(x_1) = -\frac{N}{A} K^{-1} \left\{ \int_{2a}^{\infty} \int_0^{2\pi} H^1(n) \, d\theta \, dr + \int_{2a}^{\infty} \int_0^{2\pi} H^2(n) \, d\theta \, dr \right\} : \varepsilon^{**} - \frac{N}{A} \left\{ \int_{2a}^{\infty} \int_0^{2\pi} [L(n) \cdot H^1(n)] \, d\theta \, dr \right\} : \varepsilon^{**} + \cdots \tag{33}
\]

where \( n \) is the normal vector (i.e., \( n = r/r \) with \( r = x_2 - x_1 \)). In addition, the following identities can be easily derived:

\[
\int_0^{2\pi} n_i n_j \, d\theta = \pi \delta_{ij} \tag{34}
\]

\[
\int_0^{2\pi} n_i n_j n_k n_l \, d\theta = \frac{\pi}{4} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \tag{35}
\]

It is straightforward to verify that the integrals of \( H^1 \) and \( H^2 \) in the first line of (33) are identically zero.

The (approximate) ensemble-area averaged eigenstrain tensor can now be obtained by substituting the expressions for \( L \) and \( H^1 \) into the integral of (33) and utilizing the identities (34)–(35), together with the definition of \( \tilde{\varepsilon}^{**} \) in (14). The final expression (based on (32)) reads:

\[
\langle \tilde{\varepsilon}^* \rangle = \Gamma : \varepsilon^{**} \tag{36}
\]
where the components of the isotropic tensor $\Gamma$ are rendered by

$$\Gamma_{ijkl} = \gamma_1 \delta_{ij} \delta_{kl} + \gamma_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$  \hspace{1cm} (37)$$

in which

$$\gamma_1 = \frac{\phi}{4\beta^2} \left[ -2 + \frac{\beta(1 - 2\nu_0)}{\alpha + \beta} \right]$$ \hspace{1cm} (38)

$$\gamma_2 = \frac{1}{2} + \frac{\phi}{4\beta^2} \left[ 2 + \frac{\beta(1 - 2\nu_0)}{\alpha + \beta} \right]$$ \hspace{1cm} (39)$$

It is noticed that, in deriving (36), the ensemble average $\langle \bar{\mathbf{d}}^* \rangle_{x_i}$ is a constant for any particle centered at $x_i$ — a consequence of (22) or (17). Clearly, Eqn. (36) is a truly closed-form analytical formula.

**Case II: General Radial Distribution Function**

For a general 2-D transversely isotropic radial distribution function $g(r)$ (which depends on the fiber volume fraction $\phi$), the ensemble integration of (30) leads to

$$\langle \bar{\mathbf{d}}^* \rangle = \frac{N}{A} \int_{2a}^{\infty} r g(r) \left[ \int_0^{2\pi} \bar{\mathbf{d}}^* d\theta \right] dr$$ \hspace{1cm} (40)

or

$$\langle \bar{\mathbf{d}}^* \rangle = \frac{N}{A} \frac{2\pi}{\beta^2} W : \epsilon^{*o} \times \int_{2a}^{\infty} \frac{a^4}{r^3} g(r) dr$$ \hspace{1cm} (41)$$

where

$$W_{ijkl} \equiv \xi_1 \delta_{ij} \delta_{kl} + \xi_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$ \hspace{1cm} (42)

$$\xi_1 = -2 + \frac{\beta(1 - 2\nu_0)}{\alpha + \beta}$$ \hspace{1cm} (43)

$$\xi_2 = 2 + \frac{\beta(1 - 2\nu_0)}{\alpha + \beta}$$ \hspace{1cm} (44)$$

Furthermore, we have

$$\int_{2a}^{\infty} \frac{a^4}{r^3} g(r) dr = a^2 \int_0^{\frac{\beta}{a}} \rho g(\rho) d\rho \equiv a^2 Y(g)$$ \hspace{1cm} (45)$$

Therefore, $\langle \bar{\mathbf{d}}^* \rangle$ and $\Gamma$ can be rephrased, respectively, as

$$\langle \bar{\mathbf{d}}^* \rangle = \frac{2\phi}{\beta^2} Y(g) W : \epsilon^{*o}$$ \hspace{1cm} (46)$$

$$\Gamma = I + \frac{2\phi}{\beta^2} Y(g) W \equiv \gamma_1(g) I \otimes I + 2\gamma_2(g) I$$ \hspace{1cm} (47)$$
where

$$\gamma_1(g) = \frac{2\phi}{\beta^2} Y(g) \left[ -2 + \frac{\beta(1-2v_0)}{\alpha + \beta} \right]$$  \hspace{1cm} (48)

$$\gamma_2(g) = \frac{1}{2} + \frac{2\phi}{\beta^2} Y(g) \left[ 2 + \frac{\beta(1-2v_0)}{\alpha + \beta} \right]$$ \hspace{1cm} (49)

Consequently, the "interaction-effect tensor" $\mathbf{T}$ can be explicitly computed for any 2-D transversely isotropic radial distribution function at any specified fiber volume fraction $\phi$.

For example, at higher fiber volume fractions, it is sometimes assumed that the two-point conditional probability function obeys the so-called thermodynamic "equilibrium radial distribution function" (ERDF) as follows:

$$g(\hat{r}) = H(\hat{r} - 1) [1 + A(\hat{r}) \phi]$$ \hspace{1cm} (50)

where (recalling that $\hat{r} = r/2a$; cf. Hansen and McDonald (1986), Torquato and Lado (1992))

$$A(\hat{r}) = \frac{4}{\pi} \left[ \pi - 2 \sin^{-1} \left( \frac{\hat{r}}{2} \right) - \hat{r} \left( 1 - \frac{\hat{r}^2}{4} \right)^{1/2} \right] H(2 - \hat{r})$$ \hspace{1cm} (51)

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$ \hspace{1cm} (52)

The integration in Eqn. (45) for this ERDF case can be easily obtained by numerical calculation:

$$Y(g) \equiv \int_0^{\frac{1}{2}} \rho g(\rho) d\rho = \frac{1}{8} + 0.0865\phi$$ \hspace{1cm} (53)

On the other hand, for a statistically uniform radial distribution function, we have $g = 1$ and thus $Y = 1/8$. Therefore, Case I is easily recovered.
4. Effective transverse elastic moduli of two-phase composites containing unidirectionally aligned circular fibers

We now focus on the derivation of effective transverse elastic moduli of composites containing randomly located, unidirectionally aligned circular fibers. We shall employ the pairwise interaction solutions for \( \langle \varepsilon^* \rangle \) (from Section 3) and other ensemble-area averaged field equations. The proposed procedure can be readily modified to include circular fibers of different sizes and/or elastic properties. In what follows, angle brackets denoting the ensemble-average operators will be dropped for simplicity.

In accord with Ju and Chen (1994a) and Zhao, Tandon and Weng (1989), we have the following relations governing the averaged stress \( \langle \sigma \rangle \), the averaged strain \( \langle \varepsilon \rangle \), the uniform remote strain \( \varepsilon^o \) and the averaged eigenstrain \( \varepsilon^* \):

\[
\langle \sigma \rangle = C_0 : (\varepsilon - \phi \varepsilon^*) \tag{54}
\]

\[
\varepsilon = \varepsilon^o + \phi \mathbf{s} : \varepsilon^* \tag{55}
\]

Upon substitution of the solution of \( \varepsilon^* \) in (36) or (47) into (55) and utilizing the relation between \( \varepsilon^o \) and \( \varepsilon^* \) given by (6), the relation between the averaged eigenstrain \( \varepsilon^* \) and the averaged strain \( \varepsilon \) is expressed as

\[
\varepsilon^* = \left[ \mathbf{I} \cdot (-A - s + \phi s \cdot \mathbf{I})^{-1} \right] : \varepsilon \tag{56}
\]

The above expression is valid for any 2-D transversely isotropic radial distribution function \( g(r) \).

Moreover, substitution of (56) into (54) renders the effective stiffness \( C^* \) relating \( \sigma \) and \( \varepsilon \):

\[
C^* = C_0 \cdot \left\{ \mathbf{I} - \phi \mathbf{I} \cdot (-A - s + \phi s \cdot \mathbf{I})^{-1} \right\} \tag{57}
\]

Since all the fourth-rank tensors on the right-hand side of (57) are isotropic in two-dimension, the effective stiffness tensor \( C^* \) for this two-phase composite is also isotropic in 2-D (or, equivalently, transversely isotropic in three-dimension). Accordingly, the effective plane-strain bulk modulus \( \kappa^* \) and shear modulus \( \mu_T^* \) can be explicitly evaluated:

\[
\kappa^* = \kappa_0 \left\{ 1 + \frac{8\phi (1 - \nu_0)(\gamma_1 + \gamma_2)}{(\alpha + \beta) - 4\phi(\gamma_1 + \gamma_2)} \right\} \tag{58}
\]

\[
\mu_T^* = \mu_0 \left\{ 1 + \frac{8\phi (1 - \nu_0)\gamma_2}{\beta - 2(3 - 4\nu_0)\phi \gamma_2} \right\} \tag{59}
\]
It should be noted that the definition of the effective plane-strain bulk modulus is \( K^* \equiv \lambda^* + \mu_T^* \), where \( \lambda^* \) and \( \mu_T^* \) are the effective Lamé constants. In addition, \( \gamma_1 \) and \( \gamma_2 \) are previously defined by (48)–(49). In particular, we have

\[
Y(g) = \begin{cases} 
\frac{1}{8} & \text{for uniform radial distribution} \\
\frac{1}{8} + 0.0865\phi & \text{for equilibrium radial distribution}
\end{cases}
\]  

(60)

We shall now consider some interesting special cases.

**Case I: Noninteracting Solution.** If near-field pairwise fiber interactions are totally neglected, we obtain the so-called “noninteracting” approximation for effective transverse elastic properties. The noninteracting solution can be easily acquired by dropping the pairwise interaction effects; i.e., let \( J = I \) with \( \gamma_1 = 0 \) and \( \gamma_2 = 1/2 \). Accordingly, the plane-strain effective bulk modulus \( \kappa^* \) and transverse shear modulus \( \mu^* \) reduce to:

\[
\begin{align*}
\kappa^* &= \kappa_0 \left( 1 + \frac{\kappa_0}{\kappa_1 - \kappa_0 + (1 - \phi) \frac{\kappa_0}{\kappa_0 + \mu_0}} \right) \\
\mu_T^* &= \mu_0 \left( 1 + \frac{\mu_0}{\mu_1 - \mu_0 + (1 - \phi) \frac{\kappa_0 + 2\mu_0}{2(\kappa_0 + \mu_0)}} \right)
\end{align*}
\]  

(61)  

(62)

It is observed that these “noninteracting” expressions are identical to the variational lower bounds (4.25) and (4.27) of Hashin (1965); see also Hill (1964) for \( \kappa^* \) bounds.

**Case II: Rigid fibers.** For an incompressible elastic matrix containing randomly located and aligned identical rigid circular fibers, the proposed interacting solution renders the following effective transverse shear modulus:

\[
\mu_T^* = \mu_0 \left( 1 + 2\phi + \frac{1 + 8Y(g)\phi}{1 - \phi - 8Y(g)\phi^2} \right)
\]  

(63)

For the special cases of uniform and equilibrium radial distribution functions (RDFs), the singularity points in (63) occur at \( \phi = 0.618 \) and 0.562, respectively. Furthermore, the Taylor’s series expansion of (63) with respect to \( \phi \) renders

\[
\mu_T^* = \mu_0 \left\{ 1 + 2\phi + 2[1 + 8Y(g)]\phi^2 + 2[1 + 16Y(g)]\phi^3 + O(\phi^4) \right\}
\]  

(64)

Eqn. (64) reduces to

\[
\mu_T^* = \mu_0 \left\{ 1 + 2\phi + 4\phi^2 + 6\phi^3 + O(\phi^4) \right\}
\]  

(65)
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for the uniform RDF, and

\[ \mu_T^* = \mu_0 \left\{ 1 + 2\phi + 4\phi^2 + 7.384\phi^3 + O(\phi^4) \right\} \]  \hspace{1cm} (66)

for the equilibrium RDF. The coefficient for the \( O(\phi) \) is 2 which is consistent with the dilute result of Eshelby (1957); see also Christensen (1993).

**Case III: Cylindrical voids.** For an incompressible matrix containing randomly located and aligned identical cylindrical voids, the proposed interacting solution leads to the following effective plane-strain bulk modulus and transverse shear modulus:

\[ \kappa^* = \mu_0 \left\{ \frac{1}{4\phi^2Y(g) + \phi} - 1 \right\} \]  \hspace{1cm} (67)

\[ \mu_T^* = \mu_0 \left\{ 1 - 2\phi \frac{1 + 10\phi Y(g)}{1 + \phi + 10\phi^2 Y(g)} \right\} \]  \hspace{1cm} (68)

5. **Some comparisons and simulations**

In order to illustrate the potential of the proposed micromechanical framework, we now compare our analytical predictions with Hill’s (1964) and Hashin’s (1965, 1972) two-point bounds, Silnutzer’s (1972) three-point bounds (cf. Milton (1982), Torquato and Lado (1992)), and limited available experimental data of Uemura et al. (1968). For demonstration purpose, we will consider the (statistically transversely isotropic) uniform and equilibrium RDFs. Following Kondo and Saito (1986), we will consider the following constituent elastic phase properties for the glass fiber reinforced epoxy matrix composite: \( E_1 = 11,660 \text{ kgf/mm}^2, \nu_1 = 0.22 \text{ (glass fiber)} \) and \( E_0 = 550 \text{ kgf/mm}^2, \nu_0 = 0.35 \text{ (epoxy resin)} \).

Figures 3 and 4 show the predicted plane-strain effective (normalized) bulk modulus \( \kappa^*/\kappa_0 \) and effective (normalized) transverse shear modulus \( \mu_T^*/\mu_0 \) of the glass-epoxy composites at various fiber volume fractions \( \phi \). We plot the theoretical predictions in Figures 3 and 4 based on Hashin’s (1965) second-order bounds, Silnutzer’s (1972) third-order bounds (with the equilibrium RDF following Torquato and Lado (1992)), and the proposed Eqn. (58)–(59) with the uniform and radial RDFs, respectively. Clearly, our analytical predictions are well within the Hashin’s (1965) two-point bounds and Silnutzer’s (1972) three-point bounds. We recall that our “noninteracting solutions” in the previous section coincide precisely with Hashin’s (1965) lower bounds for \( \kappa^* \) and \( \mu_T^* \) of fiber reinforced elastic composites.
The proposed micromechanical plane strain framework cannot predict the effective axial (out-of-plane) Young’s modulus $E_A^*$ and effective axial Poisson’s ratio $\nu_A^*$. On the other hand, bounds on effective axial $E_A^*$ and $\nu_A^*$ are available from Hashin (1972):

$$
E_A^* = E_0 \phi_0 + E_1 \phi + \frac{4 \phi \phi_0 (v_1 - v_0)^2}{\phi_0 \phi + \frac{1}{\kappa_0} + \frac{1}{\mu_0}} 
$$

(69)

$$
\nu_A^* = \nu_0 \phi_0 + \nu_1 \phi + \frac{\phi \phi_0 (v_1 - v_0) \left( \frac{1}{\kappa_0} - \frac{1}{\kappa_1} \right)}{\phi_0 \phi + \frac{1}{\kappa_0} + \frac{1}{\mu_0}} 
$$

(70)

where $\phi_0 \equiv 1 - \phi$. According to the calculations of Kondo and Saito (1986) and present authors, Eqn. (69) and (70) render highly accurate predictions because the lower and upper bounds of Hashin (1972) are extremely close to each other. Even the simple mixture rule (the first two terms on the right-hand side of (69)–(70)) provides fairly good estimates for effective axial $E_A^*$ and $\nu_A^*$. Therefore, the out-of-plane fiber interaction effects are insignificant as far as $E_A^*$ and $\nu_A^*$ are concerned.

Consequently, the effective transverse Young’s modulus $E_T^*$ and Poisson’s ratio $\nu_T^*$ can be predicted by combining our Eqn. (58)–(59) for $\kappa^*$ and $\mu_T^*$ and (69)–(70) for $E_A^*$ and $\nu_A^*$:

$$
E_T^* = \frac{4 \kappa^* \mu_T^*}{\kappa^* + \psi \mu_T^*} 
$$

(71)

$$
\nu_T^* = \frac{\kappa^* - \psi \mu_T^*}{\kappa^* + \mu_T^*} 
$$

(72)

where

$$
\psi = 1 + \frac{4 \nu_A^2 \kappa^*}{E_A^*} 
$$

(73)

The above expressions were given by Hashin and Rosen (1964); see Eqn. (17)–(18) therein. Figure 5 depicts the predictions for the effective transverse Young’s modulus $E_T^*$ according to the proposed Eqn. (58)–(59) and (71), and the Hashin’s (1972) bounds. Both the uniform and equilibrium RDFs are employed to illustrate the proposed model. In addition, the experimental data of Uemura et al. (1968) for the glass-epoxy composites are plotted in Figure 5 for comparison purpose. It is observed that the proposed formulation compares very well with experimental data for $\phi$ up to about 55%. For fiber volume fraction $\phi$ greater than 60%, micro-defects may exist widely in experimental specimens. Therefore, interface debonding as well as fracture may significantly affect the overall mechanical behavior of composites at very dense fiber concentrations.
Furthermore, higher-order fiber interactions would need to be considered for $\phi \geq 60\%$ by means of rigorous micromechanics and the ensemble-area averaging procedure.

It is also interesting to illustrate the potential of the proposed method in predicting the effective elastic moduli of brittle elastic matrix containing randomly located yet unidirectionally aligned cylindrical voids. In such event, we simply have $\kappa_1 = 0$ and $\mu_1 = 0$ for the inclusion phase. We shall assume that $E_0 = 0.75 \times 10^6$ bars and $\nu_0 = 0.23$ for the glass matrix. Figure 6 compares the analytical predictions of the effective (normalized) plane-strain bulk modulus $\kappa^*/\kappa_0$ produced by the Hashin's (1965) upper bound, Silnutzer's (1972) 3-point upper bound, and the proposed method (by using the uniform and equilibrium RDFs). No available experimental data are found at this time for comparison.

As indicated by Christensen (1993), the manufacturing operations for fiber composite materials often involve the flow behavior of the composite system as viscous fluid suspensions. The matrix phase is usually treated as incompressible in its fluid state, and the aligned fibers are treated as rigid (in comparison with the matrix). Therefore, the (rheological) effective transverse shear viscosity $\eta_T^*$ of these composite melts can be represented by the proposed Eqn. (63), with $\mu_T^*$ and $\mu_0$ replaced by $\eta_T^*$ and $\eta_0$. Figure 7 compares the theoretical predictions from Hashin's (1965) lower bound, Silnutzer's (1972) three-point lower bound (with the equilibrium RDF), Christensen's (1990, 1993) generalized self-consistent method (GSCM), and the proposed micromechanical interaction formulation (with the uniform and equilibrium RDFs). We observe that significant differences exist between our predictions and the other two bounds for $\phi$ greater than 30%. No experimental data are available in the open literature at this time for us to compare the analytical predictions. However, Ju and Chen (1994b) presented detailed experimental comparisons against the authors' micromechanical interaction formulation, the three-point lower bounds, and other methods for the effective shear viscosity vs. the particle volume fraction $\phi$ of colloidal suspensions containing an incompressible fluid matrix and randomly dispersed spherical rigid particles. Ju and Chen (1994b, Figure 6) showed that their micromechanical interaction formulation performed quite well while the three-point bound predictions did not fare well for $\phi$ greater than 30% in the colloidal suspensions. Although different in values, Christensen's (1990, 1993) predictions exhibit similar trend as our analytical predictions concerning effective shear viscosities $\eta_T^*$ vs. $\phi$ of circular fibers or spherical particles.
6. Conclusion

Based on the governing micromechanical field equations and the approximate pairwise fiber interaction solutions, a new micromechanical ensemble-area average approach is presented to predict effective transverse elastic moduli of linear two-phase composites containing randomly located yet aligned circular fibers. The proposed framework can be modified to accommodate circular fibers of different sizes and/or elastic properties. The ensemble-area averaged eigenstrains in fibers are approximately evaluated by Eqn. (36) or (47) through the pairwise inter-fiber interactions. Hence, a compact analytical formula (57) is derived. The proposed closed-form predictions are compared with Hashin’s (1965) two-point bounds, Silnutzer’s (1972) three-point bounds, and some available experimental data. These comparisons and simulations encompass fiber reinforced elastic composites, elastic matrix with randomly located cylindrical voids, and aligned viscous fiber suspensions. No Monte Carlo simulations nor finite element calculations are needed in the proposed framework.

The authors are currently working on the extension of the proposed method to predict the effective elastoplastic behavior of two-phase ductile matrix composites containing unidirectionally aligned yet randomly located elastic fibers. The methods proposed by Ju and Chen (1994c) and Ju and Tseng (1996a, 1996b) will be adopted to provide micromechanical ensemble-area averaged estimates.

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7. References


Effective Transverse Elastic Moduli of Fiber Reinforced Composites


8. Appendix I: Plane strain Eshelby's tensor for a circular inclusion

According to the Eshelby's solution, the elastic displacement field due to inclusion in an isotropic infinite body reads

\[ u_i(x) = -C_{jkmn}e^*_m \int G_{ij,k}(x-x') \, dx' \]  

(74)

where the second rank plane-strain Green's function is given by Mura (1987):

\[ G_{ij}(x-x') = \frac{1}{8\pi(1-\nu_0)\mu_0} \left[ \frac{(x_i-x'_i)(x_j-x'_j)}{||x-x'||^2} - (3-4\nu_0) \delta_{ij} \ln||x-x'|| \right] \]  

(75)

By taking the derivative of \( G_{ij}(x-x') \) in Eqn. (75) with respect to \( x_k \) and substituting the result into Eqn. (74), we arrive at

\[ u_i(x) = -\frac{\varepsilon^*_{jk}}{4\pi(1-\nu_0)} \int_0^{2\pi} g_{ijk}(l) \frac{dx'}{||x'-x||} \]  

(76)

where

\[ g_{ijk}(l) = (1-2\nu_0) (\delta_{ij}l_k + \delta_{ik}l_j - \delta_{jk}l_i) + 2 l_i l_j l_k \]  

(77)

and \( 1 \equiv (x'-x)/||x'-x|| \). When the point \( x \) is located inside the inclusion, the strain and stress fields become uniform for the interior points. Moreover, Eqn. (76) can be integrated explicitly. The differential element \( dx' \) can be written as

\[ dx' = r \, dr \, d\theta \]  

(78)

where \( r = ||x'-x|| \) and \( d\theta \) is the differential angle element centered at point \( x(x_1, x_2) \); see Figure 8 for a schematic plot of a circular inclusion. Upon integration with respect to \( r \), Eqn. (76) becomes

\[ u_i(x) = -\frac{\varepsilon^*_{jk}}{4\pi(1-\nu_0)} \int_0^{2\pi} r(l)g_{ijk}(l) \, d\theta \]  

(79)

Here, \( r(l) \) is the positive root of the following equation

\[ (x_1 + rl_1)^2 + (x_2 + rl_2)^2 = a^2 \]  

(80)

Therefore, we have

\[ r(l) = \frac{f}{h} + \sqrt{\frac{f^2}{h^2} + \frac{e}{h}} \]  

(81)

where

\[ h = \frac{l_1^2 + l_2^2}{a^2} \]  

(82)
Substituting Eqn. (81) into Eqn. (79), we find that the integration involving the $\sqrt{f^2/h^2 + e/h}$ term vanishes because it is even in $l$ while $g_{ijk}$ is odd in $l$. Consequently, we obtain

$$ u_i(x) = \frac{x_j \epsilon_{mn}^*}{4\pi(1-\nu_0)} \int_0^{2\pi} \frac{\lambda_j g_{imn}(l)}{h} d\theta $$

(85)

and

$$ \epsilon_{ij}(x) = \frac{\epsilon_{\text{ij}}^*}{8\pi(1-\nu_0)} \int_0^{2\pi} \frac{\lambda_i g_{jkl} + \lambda_j g_{ikl}}{h} d\theta $$

(86)

where

$$ \lambda_i \equiv l_i/a^2 $$

(87)

According to the definition of Eshelby's tensor $\epsilon_{ij} = s_{ijkl} \epsilon_{kl}^*$, we then arrive at

$$ s_{ijkl} = \frac{1}{8\pi(1-\nu_0)} \int_0^{2\pi} \frac{\lambda_i g_{jkl} + \lambda_j g_{ikl}}{h} d\theta $$

(88)

Finally, the plane-strain Eshelby's tensor for a circular inclusion can be expressed as

$$ s_{ijkl} = \frac{1}{8\pi(1-\nu_0)} \left[ (4\nu_0 - 1) \delta_{ij} \delta_{kl} + (3 - 4\nu_0) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right] $$

(89)
The product between two 2-D fourth-rank tensors $F(A_m)$ and $F(B_m)$, $m = 1$ to 6, can be shown to follow
\[ F_{ijpq}(A_m)F_{pqkl}(B_m) = F_{ijkl}(C_m) \] (90)
where $i, j, k, l, p, q = 1, 2$, $F$ is defined in (5), and
\[
\begin{align*}
C_1 &= A_1(B_1 + 4B_2 + 2B_3 + 2B_6) + 4A_2(B_1 + 2B_2 + B_3) \\
&\quad + A_4(B_1 + 2B_2 + 2B_3) + 2A_6B_1 \\
C_2 &= 2A_2(B_2 + B_6) + 2A_6B_2 \\
C_3 &= A_3(B_1 + 4B_2 + 2B_3 + 2B_6) + A_5(B_1 + 4B_2 + 2B_3) + 2A_6B_3 \\
C_4 &= A_1(B_4 + B_5) + 4A_2(B_4 + B_5) + A_4(B_4 + 2B_5 + 2B_6) + 2A_6B_4 \\
C_5 &= A_3(B_4 + B_5) + A_5(B_4 + 2B_5 + 2B_6) + 2A_6B_5 \\
C_6 &= 2A_6B_6
\end{align*}
\] (91-96)

One should note that the tensorial product of $F(A_m)$ and $F(B_m)$ does not commute in general; i.e.,
\[ F(A_m) \cdot F(B_m) \neq F(B_m) \cdot F(A_m) \] (97)

To find the inverse of $F$, we first recall the definition of the fourth-rank unit tensor:
\[ I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \] (98)

Denoting now by $F(A_m)$ the inverse tensor of $F(B_m)$, we can derive the components of $F(A_m) \equiv F^{-1}(B_m)$ by solving the system of equations (91)-(96) with the following arguments in $F(C_m)$:
\[
C_1 = C_2 = C_3 = C_4 = C_5 = 0 \quad C_6 = \frac{1}{2} \] (99)

The results are
\[
\begin{align*}
A_6 &= \frac{1}{4B_6} \\
A_2 &= -\frac{B_2}{4B_6(B_2 + B_6)}
\end{align*}
\] (100, 101)

and
\[
\begin{bmatrix} A_1 \\ A_4 \end{bmatrix} = D^{-1} \begin{bmatrix} -2A_6B_1 - 4A_2(B_1 + 2B_2 + B_3) \\ -2A_6B_4 - 4A_2(B_4 + B_5) \end{bmatrix} \] (102)
\[
\begin{aligned}
\begin{pmatrix} A_1 \\ A_5 \end{pmatrix} &= D^{-1} \begin{pmatrix} -2A_6B_3 \\ -2A_6B_5 \end{pmatrix} \\
\text{where} \\
D &\equiv \begin{bmatrix} B_1 + 4B_2 + B_3 + 2B_6 & B_1 + 4B_2 + 2B_3 \\ B_4 + B_5 & B_4 + 2B_5 + 2B_6 \end{bmatrix}
\end{aligned}
\]
Figure captions

Figure 1. Schematic plot of a composite reinforced by unidirectionally aligned yet randomly located long circular fibers.

Figure 2. Schematic diagram for the plain-strain two-fiber interaction problem.

Figure 3. The effective plain-strain bulk modulus vs. the fiber volume fraction $\phi$. The upper and lower solid lines correspond to the present method with the equilibrium and uniform RDFs, respectively.

Figure 4. The effective transverse shear modulus vs. the fiber volume fraction $\phi$. The upper and lower solid lines correspond to the present method with the equilibrium and uniform RDFs, respectively.

Figure 5. The effective transverse Young’s modulus vs. the fiber volume fraction $\phi$. The upper and lower solid lines correspond to the present method with the equilibrium and uniform RDFs, respectively.

Figure 6. The effective plain-strain bulk modulus vs. the void volume fraction $\phi$ for porous materials containing aligned cylindrical voids. The lower and upper solid lines correspond to the present method with the equilibrium and uniform RDFs, respectively.

Figure 7. The effective relative transverse shear viscosity vs. the fiber volume fraction $\phi$. The upper and lower solid lines correspond to the present method with the equilibrium and uniform RDFs, respectively.

Figure 8. The domain of a fiber cross-section $\Omega$. 
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